Review

Tensor: Linear mapping of one vector onto another $\mathbf{v} = \mathbf{S}\mathbf{u}$

Tensor components (orthonormal basis) $S_{ij} = \mathbf{m}_i \cdot \mathbf{S}\mathbf{m}_j \quad v_i = S_{ij} v_j$

Dyadic Product $\mathbf{S} = \mathbf{a} \otimes \mathbf{b}$ satisfies $\mathbf{S}\mathbf{u} = (\mathbf{a} \otimes \mathbf{b})\mathbf{u} = \mathbf{a}(\mathbf{b} \cdot \mathbf{u}) \quad S_{ij} = a_i b_j$

Dyadic expansion $\mathbf{S} = S_{ij} \mathbf{m}_i \otimes \mathbf{m}_j$

Change of Basis $\mathbf{S} = S_{ij}^{(\mathbf{m})} \mathbf{m}_i \otimes \mathbf{m}_j = S_{ij}^{(\mathbf{e})} \mathbf{e}_i \otimes \mathbf{e}_j \quad S_{kl}^{(\mathbf{m})} = Q_{ki} S_{ij}^{(\mathbf{e})} Q_{lj}$

$$Q_{ij} = \mathbf{m}_i \cdot \mathbf{e}_j$$

Basic Tensor Operations

Addition: $\mathbf{U} = \mathbf{S} + \mathbf{T} \quad U_{ij} = S_{ij} + T_{ij}$

Tensor-vector products: $\mathbf{v} = \mathbf{S}\mathbf{u} \quad v_i = S_{ij} u_j \quad \mathbf{v} = \mathbf{u}\mathbf{S} \quad v_i = u_j S_{ji}$

Product of two tensors: $\mathbf{U} = \mathbf{T}\mathbf{S} \quad U_{ij} = T_{ik} S_{kj} \quad \text{NB}: \; (\mathbf{T}\mathbf{S} \neq \mathbf{S}\mathbf{T})$

Transpose: $\mathbf{S}^T \quad \mathbf{u} \cdot \mathbf{S}^T = \mathbf{S} \cdot \mathbf{u} \quad S_{ij}^T = S_{ji}$

Note: $(\mathbf{S}\mathbf{T})^T = \mathbf{T}^T \mathbf{S}^T$ (show with index notation)

Trace: $\text{trace}(\mathbf{S}) = S_{11} + S_{22} + S_{33} \quad \text{trace}(\mathbf{S}) = S_{kk}$

Contracted Products

Inner Product $T : S \equiv T_{ij} S_{ij}$

Outer Product $T \cdot\cdot S \equiv T_{ij} S_{ji} \equiv T : S^T$

Determinant

$$\lambda = \det \mathbf{A} \quad \equiv \quad \lambda = \frac{1}{6} \in_{ijk} \in_{lmn} A_{li} A_{mj} A_{nk} = \in_{ijk} A_{i1} A_{j2} A_{k3}$$

$$\Leftrightarrow \in_{lmn} \lambda = \in_{ijk} A_{li} A_{mj} A_{nk} = \in_{ijk} A_{il} A_{jm} A_{kn}$$

Useful Results

$\det(AB) = \det(A) \det(B)$

$\det(A^T) = \det(A)$

A tensor is nonsingular $\Leftrightarrow \det(S) \neq 0$

Inverse of a nonsingular tensor

$$S^{-1}S = SS^{-1} = I \qquad S^{-1}_{ij} S_{jk} = \delta_{ik}$$

$$S^{-1}_{ij} = \frac{1}{2\det(S)} \epsilon_{jpq} \epsilon_{ikl} S_{pk} S_{ql}$$

A tensor is othogonal if $S^{-1} = S^T$

is "proper" orthogonal if $S^{-1} = S^T$ and $\det(S) > 0$

Example

Let $J = \det(\mathbf{S})$. Show that $\dfrac{\partial J}{\partial S_{mn}} = J S^{-1}_{nm}$. (this is a very useful result – we often need to differentiate the determinant of a tensor when working with constitutive equations for materials, for example)

Note $$\frac{\partial S_{ij}}{\partial S_{mn}} = \delta_{im} \delta_{jn}$$

Recall $J = \frac{1}{6} \epsilon_{pqr} \epsilon_{ijk} S_{pi} S_{qj} S_{rk}$

$$\frac{\partial J}{\partial S_{mn}} = \frac{1}{6} \epsilon_{pqr} \epsilon_{ijk} \{ \delta_{pm} \delta_{in} S_{qj} S_{rk} + S_{pi} \delta_{qm} \delta_{jn} S_{rk} + S_{pi} S_{qj} \delta_{rm} \delta_{kn} \}$$

Recall $\epsilon_{ijk} = \epsilon_{jki}$

$$\frac{\partial J}{\partial S_{mn}} = \frac{1}{2} \epsilon_{mqr} \epsilon_{njk} S_{qj} S_{rk}$$

$$= \det(S)\, S^{-1}_{nm}$$

Invariants of a tensor

Definition: A scalar function of components that is independent of basis

Example: trace $(S) = S_{kk}$

$$S_{kk}^{(m)} = Q_{ki}\, S_{ij}^{e}\, Q_{kj}$$

$$= Q_{ki}\, Q_{kj}\, S_{ij}^{e} = \delta_{ij} S_{ij}^{e} = S_{ii}^{e}$$

3 independent invariants

Standard set :

$$I_1 = \text{trace}(S)$$

$$I_2 = \frac{1}{2}\left(\text{trace}(S)^2 - S \cdot\cdot S\right)$$

$$= \frac{1}{2}\left(S_{ii} S_{jj} - S_{ij} S_{ji}\right)$$

$$I_3 = \det(S)$$

Eigenvalues & Eigenvectors $\{\lambda, \underline{u}\}$

$$S\underline{u} = \lambda \underline{u}$$

To calculate λ: $(S - \lambda I)\underline{u} = \underline{0}$

$$\Rightarrow \det(S - \lambda I) = 0$$

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0$$ Characteristic equation

Observations:

(1) At least 1 real eigenvalue

(2) A symmetric tensor has real eigenvalues and orthogonal eigenvectors

Proof : Let $\{\lambda_u\ \underline{u}\}$, $\{\lambda_v, \underline{v}\}$

$$\underline{v} \cdot S\underline{u} = \lambda_u\, \underline{v}\cdot\underline{u} = \underline{u} \cdot S^T \underline{v} = \underline{u}\cdot S\underline{v}$$
$$= \lambda_v\, \underline{v}\cdot\underline{u}$$

$$(\lambda_u - \lambda_v)\, \underline{u}\cdot\underline{v} = 0 \quad \Rightarrow \underline{u}\cdot\underline{v} = 0$$

Also Let $(\lambda, \underline{u})$, $(\bar{\lambda}, \bar{\underline{u}})$ are a conjugate pair

$$\bar{\underline{u}} \cdot S\underline{u} = \lambda\, \underline{u}\cdot\bar{\underline{u}} = \underline{u}\cdot S^T \bar{\underline{u}} = \underline{u}\cdot S\bar{\underline{u}}$$
$$= \bar{\lambda}\, \bar{\underline{u}}\cdot\underline{u} \quad \Rightarrow \lambda = \bar{\lambda}$$

(3) Orthogonal tensors have eigenvalues 1 , $e^{\pm i\theta}$ for some θ

Let $R^T R = I$

Proof : $R\bar{\underline{u}} \cdot R\underline{u} = \bar{\lambda}\lambda\, \bar{\underline{u}}\cdot\underline{u}$

$= \bar{\underline{u}} \cdot R^T R \underline{u}$

$= \bar{\underline{u}} \cdot \underline{u} \Rightarrow \bar{\lambda}\lambda = 1$

Cayley - Hamilton Theorem :

Every tensor satisfies its own characteristic eq

$$S^3 - I_1 S^2 + I_2 S - I_3 = 0 \quad *$$

Proof: Let $T(\alpha)$ satisfy $T(S-\alpha I) = \det(S-\alpha I)\, I$

$$T(S-\alpha I) = (-\alpha^3 + I_1\alpha^2 - I_2\alpha + I_3)\, I$$

Guess $T = I\alpha^2 + T_2\alpha + T_3$

$$\Rightarrow T(S-\alpha I) = -\alpha^3 I + (S-T_2)\alpha^2 - (T_3 - T_2 S)\alpha + T_3 S$$
$$= -\alpha^3 I + I_1 I \alpha^2 - I_2 I \alpha + I_3 I$$

$$S - T_2 = I_1 I \qquad T_3 - T_2 S = I_2 I \qquad T_3 S = I_3 I$$

$\Rightarrow$ Subst back into *

$$S^3 - (S-T_2)S^2 + (T_3 - T_2 S)S - T_3 S = 0 \, !$$