Review

Eigenvalues and Vectors $\{\lambda, \mathbf{u}\}$: $\mathbf{S}\mathbf{u} = \lambda \mathbf{u}$

Symmetric tensors have real eigenvalues and (for non repeated eigenvalues) orthogonal eigenvectors

$$\mathbf{S}\mathbf{e}_i = \lambda_i \mathbf{e}_i \qquad \text{(no sum on } i\text{)}$$

$$\lambda_i = \bar{\lambda}_i \qquad \mathbf{e}_i \cdot \mathbf{e}_j = 0 \qquad i \neq j$$

Representations of special tensors

Symmetric $S = S^T$ Let $\{\lambda_i, \underline{e}_i\}$ satisfy

$S\underline{e}_i = \lambda_i \underline{e}_i$ (no sum on i)

$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$

Then $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ is an orthonormal basis $\leftarrow$ "principal basis"

Then $$S = \sum_{i=1}^{3} \lambda_i \, \underline{e}_i \otimes \underline{e}_i$$

or in this basis S is diagonal

This allows us to define other operations

$$S^{\frac{1}{2}} : S^{1/2} S^{1/2} = S$$

$$S^{1/2} = \sum_{i=1}^{3} \lambda_i^{1/2} \, \underline{e}_i \otimes \underline{e}_i$$

$$\log(S) = \sum_{i=1}^{3} \log(\lambda_i) \, \underline{e}_i \otimes \underline{e}_i$$

Skew Tensors $\quad W = -W^T$

(Note $\underline{u} \cdot W \underline{u} = 0$

To see this note $u_i W_{ij} u_j = -u_i W_{ji} u_j = -u_i W_{ij} u_j$)

Every skew tensor has a dual vector $\underline{n}$ such that

$$W \underline{u} = \underline{n} \times \underline{u}$$

$$n_i = -\frac{1}{2} \epsilon_{ijk} W_{jk}$$

$$W_{ij} = -\epsilon_{ijk} n_k = \epsilon_{ikj} n_k$$

Orthogonal Tensors

$$\underline{R}\,\underline{R}^T = \underline{R}^T\underline{R} = \underline{I}$$

Orthogonal tensors preserve length

$$\underline{v} = \underline{R}\,\underline{u} \qquad \underline{v}\cdot\underline{v} = \underline{u}\cdot\underline{u}$$

To see this: $(\underline{R}\,\underline{u})\cdot(\underline{R}\,\underline{u}) = \underline{u}\cdot\underline{R}^T\underline{R}\,\underline{u}$

$$= \underline{u}\cdot\underline{u}$$

Every othogonal tensor can be represented as a rotation about some axis $\underline{n}$ through some angle θ

[$\underline{n}$ is a unit vector]

$$\mathrm{trace}\,(\underline{R}) = 1 + 2\cos\theta \qquad \sin\theta\,\underline{n} = \mathrm{dual}(\underline{R} - \underline{R}^T)$$

or let W be dual$(\underset{\sim}{n})$

$$R = I + (1-\cos\theta) W W + \sin\theta \, W$$

$$R_{ij} = \delta_{ij} + (1-\cos\theta) n_i n_j + \sin\theta \, \epsilon_{ikj} n_k$$

"Rodriguez representation"

Decomposition of tensors

Skew and symmetric Let A be a tensor

Then $\exists \; S, W : A = S + W \quad S = S^T \quad W = -W^T$

$$S = \frac{1}{2}(A + A^T) \qquad W = \frac{1}{2}(A - A^T)$$

Polar Decomposition Let A be a tensor

Then $\exists$ R, U, V : $A = RU = VR$

$R^T R = R R^T = I$

$U = U^T \quad V = V^T$

To see this note $A^T A$ is symmetric.
Expand as components in principal basis

$$A^T A = \sum_{i=1}^{3} \lambda_i^2 \, \underline{e}_i \otimes \underline{e}_i$$

Let $$U = \sum_{i=1}^{3} \lambda_i \, \underline{e}_i \otimes \underline{e}_i \Leftarrow \text{symmetric}$$

Let $R = A U^{-1} \Rightarrow A = RU$

$$R^T R = (A U^{-1})^T A U^{-1}$$

$$= U^{-1} \underbrace{A^T A}_{U^2} U^{-1} = I$$

Finally let $V = R U R^T$ $\quad V = V^T$

Tensor Calculus

* Tensor Field $S(x_1, x_2, x_3)$

Calculus operations on vectors extend to tensors

* Gradient $[\nabla S]_{ijk} \equiv \dfrac{\partial S_{ij}}{\partial x_k}$

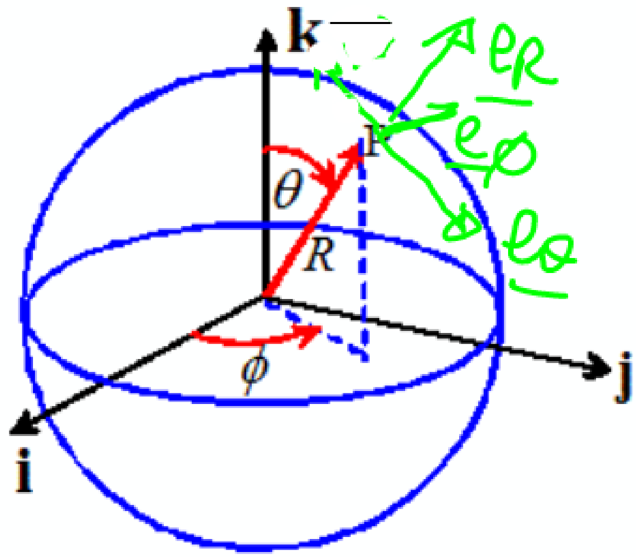
* Divergence $\quad [\nabla \cdot S]_i = \dfrac{\partial S_{ij}}{\partial x_j}$

[Some authors use $[\nabla \cdot S]_i = \dfrac{\partial S_{ji}}{\partial x_j}$]

* Divergence Theorem for tensors

$$\int_V \nabla \cdot S \, dV = \int_S S \cdot \underline{n} \, dA$$

Polar Coordinates



Let (R, θ, ϕ) be 3 coordinates

Let $\{\underline{i}, \underline{j}, \underline{k}\}$ be standard basis

$$x_1 = R\sin\theta\cos\phi$$
$$x_2 = R\sin\theta\sin\phi$$
$$x_3 = R\cos\theta$$

$$\underline{r} = R\sin\theta\cos\phi\,\underline{i} + R\sin\theta\sin\phi\,\underline{j} + R\cos\theta\,\underline{k}$$

Vectors and Tensors

Define basis vectors

$$\underline{e}_R = \frac{1}{\left|\frac{\partial \underline{r}}{\partial R}\right|}\frac{\partial \underline{r}}{\partial R} \qquad \underline{e}_\theta = \frac{1}{\left|\frac{\partial \underline{r}}{\partial \theta}\right|}\frac{\partial \underline{r}}{\partial \theta} \qquad \underline{e}_\phi = \frac{1}{\left|\frac{\partial \underline{r}}{\partial \phi}\right|}\frac{\partial \underline{r}}{\partial \phi}$$

$\{\underline{e}_R, \underline{e}_\theta, \underline{e}_\phi\}$ is orthonormal

$$\mathbf{e}_R = \sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}$$
$$\mathbf{e}_\theta = \cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k}$$
$$\mathbf{e}_\phi = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j}$$

Note

$\underline{e}_R \cdot \underline{e}_R = 1$

$\underline{e}_R \cdot \underline{e}_\theta = \underline{e}_R \cdot \underline{e}_\phi = 0$

$$\left|\frac{\partial \underline{r}}{\partial R}\right| = 1 \qquad \left|\frac{\partial \underline{r}}{\partial \theta}\right| = R \qquad \left|\frac{\partial \underline{r}}{\partial \phi}\right| = R\sin\theta$$

Now let

$$\underline{V} = V_R\,\underline{e}_R + V_\theta\,\underline{e}_\theta + V_\phi\,\underline{e}_\phi$$

$$\underline{S} = S_{RR}\,\underline{e}_R \otimes \underline{e}_R + S_{R\theta}\,\underline{e}_R \otimes \underline{e}_\theta + S_{R\phi}\,\underline{e}_R \otimes \underline{e}_\phi$$
$$+ S_{\theta R}\,\underline{e}_\theta \otimes \underline{e}_R + S_{\theta\theta}\,\underline{e}_\theta \otimes \underline{e}_\theta \quad \ldots \text{etc}$$

Or in matrix form $V = [V_R, V_\theta, V_\phi]$

$$S = \begin{bmatrix} S_{RR} & S_{R\theta} & S_{R\phi} \\ S_{\theta R} & S_{\theta\theta} & S_{\theta\phi} \\ S_{\phi R} & S_{\phi\theta} & S_{\phi\phi} \end{bmatrix}$$