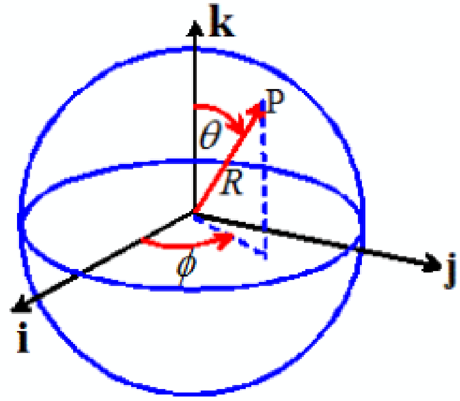
Review

Polar Coordinates (R,θ,ϕ)

Standard Basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ coordinates (x_1,x_2,x_3)

Mapping $\mathbf{r} = R\sin\theta\cos\phi\,\mathbf{i} + R\sin\theta\sin\phi\,\mathbf{j} + R\cos\theta\,\mathbf{k}$



Define normalized natural basis vectors

$$\mathbf{e}_R = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial R}\right|}\frac{\partial \mathbf{r}}{\partial R} \qquad \mathbf{e}_\theta = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \theta}\right|}\frac{\partial \mathbf{r}}{\partial \theta} \qquad \mathbf{e}_\phi = \frac{1}{\left|\frac{\partial \mathbf{r}}{\partial \phi}\right|}\frac{\partial \mathbf{r}}{\partial \phi}$$

$$\begin{aligned}\mathbf{e}_R &= \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}\\ \mathbf{e}_\theta &= \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}\\ \mathbf{e}_\phi &= -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}\end{aligned} \qquad \left|\frac{\partial \mathbf{r}}{\partial R}\right| = 1 \qquad \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = R \qquad \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = R\sin\theta$$

Vectors and Tensors

$$\mathbf{v} = v_R\mathbf{e}_R + v_\theta\mathbf{e}_\theta + v_\phi\mathbf{e}_\phi$$

$$\begin{aligned}\mathbf{S} = {}& S_{RR}\mathbf{e}_R\otimes\mathbf{e}_R + S_{R\theta}\mathbf{e}_R\otimes\mathbf{e}_\theta + S_{R\phi}\mathbf{e}_R\otimes\mathbf{e}_\phi +\\ & S_{\theta R}\mathbf{e}_\theta\otimes\mathbf{e}_R + S_{\theta\theta}\mathbf{e}_\theta\otimes\mathbf{e}_\theta + S_{\theta\phi}\mathbf{e}_R\otimes\mathbf{e}_\phi +\\ & S_{\phi R}\mathbf{e}_\phi\otimes\mathbf{e}_R + S_{\phi\theta}\mathbf{e}_\phi\otimes\mathbf{e}_\theta + S_{\phi\phi}\mathbf{e}_\phi\otimes\mathbf{e}_\phi\end{aligned}$$

Basis Change Formulas

$\underline{e}_i^c \quad V_i^c \quad S_{ij}^c$ – Cartesian $\{\underline{i}, \underline{j}, \underline{k}\}$ components

$\underline{e}_i^p \quad V_i^p \quad S_{ij}^p$ – Polar components

Recall $\quad V_i^p = Q_{ij} V_j^c \qquad S_{ij}^p = Q_{ik} S_{kl}^c Q_{jl}$

$$Q_{ij} = \underline{e}_i^p \cdot \underline{e}_j^c$$

$Q_{11} = \sin\theta\cos\phi \qquad Q_{12} = \sin\theta\sin\phi \qquad Q_{13} = \cos\theta$

etc ....

$$\begin{bmatrix} a_R \\ a_\theta \\ a_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} a_R \\ a_\theta \\ a_\phi \end{bmatrix}$$

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$$\begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{xy} & S_{zz} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} S_{RR} & S_{R\theta} & S_{R\phi} \\ S_{\theta R} & S_{\theta\theta} & S_{\theta\phi} \\ S_{\phi R} & S_{\phi\theta} & S_{\phi\phi} \end{bmatrix} \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}$$

$$\begin{bmatrix} S_{RR} & S_{R\theta} & S_{R\phi} \\ S_{\theta R} & S_{\theta\theta} & S_{\theta\phi} \\ S_{\phi R} & S_{\phi\theta} & S_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{xy} & S_{zz} \end{bmatrix} \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}$$

Calculus in polar coordinates

Preliminary:- Need derivatives of basis vecs

Recall, eg $\underline{e}_R = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}$

$\Rightarrow \dfrac{\partial \underline{e}_R}{\partial R} = 0$; $\dfrac{\partial \underline{e}_R}{\partial \theta} = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}$
$= \underline{e}_\theta$

$$\frac{\partial \mathbf{e}_R}{\partial R} = \frac{\partial \mathbf{e}_\theta}{\partial R} = \frac{\partial \mathbf{e}_\phi}{\partial R} = 0 \qquad \frac{\partial \mathbf{e}_R}{\partial \theta} = \mathbf{e}_\theta \qquad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_R \qquad \frac{\partial \mathbf{e}_\phi}{\partial \theta} = 0$$

$$\frac{\partial \mathbf{e}_R}{\partial \phi} = \sin\theta \mathbf{e}_\phi \qquad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta \mathbf{e}_\phi \qquad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin\theta \mathbf{e}_R - \cos\theta \mathbf{e}_\theta$$

Gradient Operator

Let $f(R, \theta, \phi)$ be a scalar field

$$df = \frac{\partial f}{\partial R} dR + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi = [\nabla f] \cdot d\underline{r}$$

Note also $d\underline{r} = \frac{\partial \underline{r}}{\partial R} dR + \frac{\partial \underline{r}}{\partial \theta} d\theta + \frac{\partial \underline{r}}{\partial \phi} d\phi$

$$= \underline{e}_R \, dR + R \underline{e}_\theta \, d\theta + R\sin\theta \, \underline{e}_\phi \, d\phi$$

$$\Rightarrow \quad dR = \underline{e}_R \cdot d\underline{r} \qquad d\theta = \frac{1}{R} \underline{e}_\theta \cdot d\underline{r} \qquad d\phi = \frac{\underline{e}_\phi \cdot d\underline{r}}{R\sin\theta}$$

Hence

$$df = \underbrace{\left(\underline{e}_R \frac{\partial}{\partial R} + \frac{\underline{e}_\theta}{R} \frac{\partial}{\partial \theta} + \frac{\underline{e}_\phi}{R\sin\theta} \frac{\partial}{\partial \phi} \right)}_{\nabla} f \cdot d\underline{r}$$

All other operations $\nabla \underline{V}$, $\nabla \cdot \underline{V}$, $\nabla \cdot S$ etc follow from this

Example : Let $\underline{V} = V_r(R)\, \underline{e}_R$

Find $\nabla \underline{V}$

$$= \underbrace{\left(\underline{e}_R \frac{\partial}{\partial R} + \frac{\underline{e}_\theta}{R} \frac{\partial}{\partial \theta} + \frac{\underline{e}_\phi}{R\sin\theta} \frac{\partial}{\partial \phi} \right) V_r(R)\, \underline{e}_R}_{\text{Become dyadic products}}$$

$$= \frac{\partial (V_r \underline{e_R})}{\partial R} \otimes \underline{e_R} + \frac{1}{R}\frac{\partial}{\partial \theta}(V_r \underline{e_R}) \otimes \underline{e_\theta} + \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}(V_r(R)\underline{e_R}) \otimes \underline{e_\phi}$$

$$= \frac{\partial V_r}{\partial R}\underline{e_R} \otimes \underline{e_R} + \frac{V_r}{R}\underline{e_\theta} \otimes \underline{e_\theta} + \frac{V_r}{R}\underline{e_\phi} \otimes \underline{e_\phi}$$

More complicated vectors follow with same process

Gradient of a scalar

$$\nabla f = \left(\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\theta \frac{1}{R}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi} \right) f = \mathbf{e}_R \frac{\partial f}{\partial R} + \mathbf{e}_\theta \frac{1}{R}\frac{\partial f}{\partial \theta} + \mathbf{e}_\phi \frac{1}{R\sin\theta}\frac{\partial f}{\partial \phi}$$

Gradient of a vector

$$\nabla \mathbf{v} = \left(v_R \mathbf{e}_R + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi \right) \otimes \left(\mathbf{e}_R \frac{\partial}{\partial R} + \mathbf{e}_\theta \frac{1}{R}\frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi} \right)$$

$$\nabla \mathbf{v} \equiv \begin{bmatrix} \frac{\partial v_R}{\partial R} & \frac{1}{R}\frac{\partial v_R}{\partial \theta} - \frac{v_\theta}{R} & \frac{1}{R\sin\theta}\frac{\partial v_R}{\partial \phi} - \frac{v_\phi}{R} \\ \frac{\partial v_\theta}{\partial R} & \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R}{R} & \frac{1}{R\sin\theta}\frac{\partial v_\theta}{\partial \phi} - \cot\theta \frac{v_\phi}{R} \\ \frac{\partial v_\phi}{\partial R} & \frac{1}{R}\frac{\partial v_\phi}{\partial \theta} & \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \cot\theta \frac{v_\theta}{R} + \frac{v_R}{R} \end{bmatrix}$$

Divergence of a vector

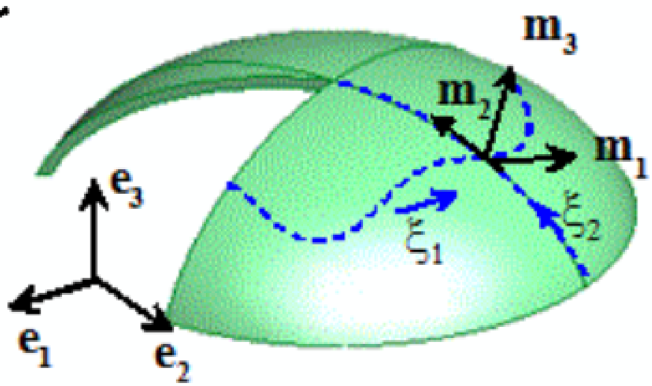
$$\nabla \cdot \mathbf{v} \equiv trace(\nabla \mathbf{v}) = \frac{\partial v_R}{\partial R} + 2\frac{v_R}{R} + \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \cot\theta \frac{v_\theta}{R}$$

Divergence of a tensor

$$\nabla \cdot \mathbf{S} \equiv \begin{bmatrix} \frac{\partial S_{RR}}{\partial R} + 2\frac{S_{RR}}{R} + \frac{1}{R}\frac{\partial S_{\theta R}}{\partial \theta} + \cot\theta \frac{S_{\theta R}}{R} + \frac{1}{R\sin\theta}\frac{\partial S_{\phi R}}{\partial \phi} - \frac{1}{R}\left(S_{\theta\theta} + S_{\phi\phi}\right) \\ \frac{\partial S_{R\theta}}{\partial R} + 2\frac{S_{R\theta}}{R} + \frac{1}{R}\frac{\partial S_{\theta\theta}}{\partial \theta} + \cot\theta \frac{S_{\theta\theta}}{R} + \frac{1}{R\sin\theta}\frac{\partial S_{\phi\theta}}{\partial \phi} + \frac{S_{\theta R}}{R} - \cot\theta \frac{S_{\phi\phi}}{R} \\ \frac{\partial S_{R\phi}}{\partial R} + 2\frac{S_{R\phi}}{R} + \frac{\sin\theta}{R}\frac{\partial S_{\theta\phi}}{\partial \theta} + \cos\theta \frac{S_{\theta\phi}}{R} + \frac{1}{R\sin\theta}\frac{\partial S_{\phi\phi}}{\partial \phi} + \frac{1}{R}\left(S_{\phi R} + S_{\phi\theta}\right) \end{bmatrix}$$

Overview of general curvilinear coordinates

Let $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$ be standard basis



$\underline{r} = x_i \underline{e}_i$

Introduce (ξ_1, ξ_2, ξ_3)

Let $x_i(\xi_1, \xi_2, \xi_3)$ be an invertible 1:1 mapping $\xi_i \leftrightarrow x_i$

Define "natural Basis" (covariant basis vectors)

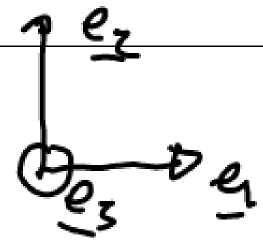
$$\underline{m}_i = \frac{\partial \underline{r}}{\partial \xi_i}$$

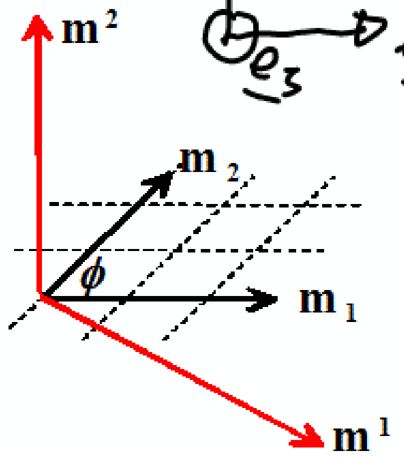
Also "reciprocal Basis" (contravariant basis) $\underline{m}^i$

such that

$$\underline{m}^i \cdot \underline{m}_j = \delta^i_j \qquad \left(\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \right)$$

eg $\underline{m}^1 = \dfrac{\underline{m}_2 \times \underline{m}_3}{\underline{m}_1 \cdot \underline{m}_2 \times \underline{m}_3}$ $\quad \underline{m}^2 = \dfrac{\underline{m}_3 \times \underline{m}_1}{\underline{m}_2 \cdot \underline{m}_3 \times \underline{m}_1}$ etc





Example : Sheared coordinates

$$x_1 = \xi_1 + \alpha \xi_2 \qquad \alpha = \cos \phi$$

$$x_2 = \beta \xi_2 \qquad \beta = \sin \phi$$

$$x_3 = \xi_3$$

$$\underline{m}_1 = \underline{e}_1 \qquad \underline{m}_2 = \alpha \underline{e}_1 + \beta \underline{e}_2 \qquad \underline{m}_3 = \underline{e}_3$$

$$\underline{m}^1 = \underline{e}_1 - \frac{\alpha}{\beta}\underline{e}_2 \qquad \underline{m}^2 = \frac{\underline{e}_2}{\beta} \qquad \underline{m}^3 = \underline{e}_3$$

Vectors and Tensors

We can use any of our basis vectors 🙂

$$\underline{V} = \underbrace{V^i}_{\text{contravariant components}} \underline{m}_i = \underbrace{V_i}_{\text{covariant components}} \underline{m}^i$$

$$\underline{S} = S^{ij} \underline{m}_i \otimes \underline{m}_j = S_{ij} \underline{m}^i \otimes \underline{m}^j = S_i{}^j \underline{m}^i \otimes \underline{m}_j = S^i{}_j \underline{m}_i \otimes \underline{m}^j$$