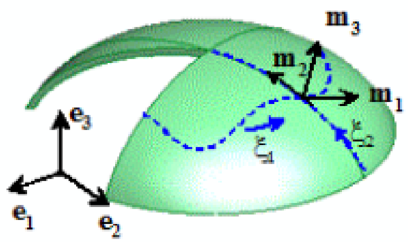
Review: Curvilinear Coordinates



Standard Basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

Point $\mathbf{r} = x_i \mathbf{e}_i$

Curvilinear Coordinates (ξ_1, ξ_2, ξ_3)

Invertible 1:1 mapping $x_i = x_i(\xi_1, \xi_2, \xi_3)$

Natural (covariant) basis $\mathbf{m}_i = \dfrac{\partial \mathbf{r}}{\partial \xi_i}$

Reciprocal (contravariant) basis $\mathbf{m}^j$ $\quad \mathbf{m}_i \cdot \mathbf{m}^j = \delta_i^j$

Vectors and tensors $\mathbf{v} = v^i \mathbf{m}_i = v_i \mathbf{m}^i$

$$\mathbf{S} = S^{ij} \mathbf{m}_i \otimes \mathbf{m}_j = S_{ij} \mathbf{m}^i \otimes \mathbf{m}^j = S_i^{\cdot j} \mathbf{m}^i \otimes \mathbf{m}_j = S^i_{\cdot j} \mathbf{m}_i \otimes \mathbf{m}^j$$

Note that $v^k = \underline{v} \cdot \underline{m}^k \qquad v_k = \underline{v} \cdot \underline{m}_k$

To see this $v^i \underline{m}_i = \underline{v} \Rightarrow \underbrace{\underline{m}^k \cdot v^i \underline{m}_i}_{\delta^k_i} = \underline{m}^k \cdot \underline{v}$

$$v^k = \underline{m}^k \cdot \underline{v}$$

Similarly $S^{ij} = \underline{m}^i \cdot S\, \underline{m}^j \qquad S_{ij} = \underline{m}_i \cdot S\, \underline{m}_j$

Metric Tensor

$$g = \underline{m}^i \otimes \underline{m}_i = \underline{m}_i \otimes \underline{m}^i = \underline{m}_i \otimes \delta^k_i\, \underline{m}^k$$

$$g_{ij} = \underline{m}_i \cdot g\, \underline{m}_j = \underline{m}_i \cdot \underline{m}^k \otimes \underline{m}_k \cdot \underline{m}_j = \underline{m}_i \cdot \underline{m}_j$$

$$g^{ij} = \underline{m}^i \cdot \underline{m}^j$$

Properties of g

$$g\,g = (\underline{m}_i \otimes \underline{m}^i)(\underline{m}_k \otimes \underline{m}^k) = \underline{m}_i \otimes \underline{m}^i = g$$

* g is the identity tensor!

Note also

$$g\,\underline{v} = g_{ij}\, \underline{m}^i \otimes \underline{m}^j \, v^k \underline{m}_k$$
$$= g_{ij}\, v^j \underline{m}^i$$
$$= v_i\, \underline{m}^i$$

$$v_i = g_{ij} v^j \qquad S_{ij} = g_{ik} S^{kl} g_{lj}$$

$$v^i = g^{ij} v_j$$

Use g^{ij} and g_{ij} to raise / lower indices

Vector Operation

Dot Product

$$\underline{a} \cdot \underline{b} = a^i \underline{m}_i \cdot b_j \underline{m}^j = a^i b_i$$

$$= a_i b^i$$

Cross product

Define $\quad e_{ijk} = \underline{m}_i \cdot (\underline{m}_j \times \underline{m}_k) = \epsilon_{ijk} \sqrt{\det(g_{ij})}$

$$e^{ijk} = \underline{m}^i \cdot (\underline{m}^j \times \underline{m}^k) = \epsilon_{ijk} / \sqrt{\det(g_{ij})}$$

$$\underline{c} = \underline{a} \times \underline{b}$$

$$c_\ell \underline{m}^\ell = a^j \underline{m}_j \times b^k \underline{m}_k$$

$$\underline{m}_i \cdot c_\ell \underline{m}^\ell = a^j b^k \underline{m}_i \cdot [\underline{m}_j \times \underline{m}_k]$$

$$c_i = e_{ijk}\, a^j\, b^k$$

$$c^i = e^{ijk}\, a_j\, b_k$$

Product of a tensor & vector

$$\underline{u} = S\,\underline{v} = S^{ij}\, \underline{m}_i \times \underline{m}_j \quad v_k\, \underline{m}^k$$

$$= S^{ij}\, v_j\, \underline{m}_i$$

$$u^i = S^{ij}\, v_j$$

$$u_i = S_i^{\ j}\, v_j = S_{ij}\, v^j$$

Calculus

Preliminary : Basis Vector Derivatives

Recall $\underline{m}_i = \dfrac{\partial \underline{r}}{\partial \xi_i}$

$$d\underline{m}_i = \frac{\partial^2 \underline{r}}{\partial \xi_i \partial \xi_j} d\xi_j$$

Express $d\underline{m}_i$ as components in $\underline{m}_i$

Define $\Gamma^k_{ij} = \underline{m}^k \cdot \dfrac{\partial^2 \underline{r}}{\partial \xi_i \partial \xi_j}$ ⇐ Christoffel Symbol of 2^{nd} kind

$$d\underline{m}_i = \Gamma^k_{ij} \, d\xi_j \, \underline{m}_k$$

Gradient Operator ∇

Consider scalar function $f(\xi_1, \xi_2, \xi_3)$

$$df = \frac{\partial f}{\partial \xi_j} d\xi_j = (\nabla f) \cdot d\underline{r}$$

Recall $\underline{r}$ is a given function of ξ

$$d\underline{r} = \frac{\partial \underline{r}}{\partial \xi_k} d\xi_k = \underline{m}_k d\xi_k$$

$$\Rightarrow \underline{m}^i \cdot d\underline{r} = d\xi_i$$

Covariant derivative

$$\Rightarrow df = \underbrace{\left(\underline{m}^j \frac{\partial}{\partial \xi^j} \right)}_{\text{gradient operator}} f \cdot d\underline{r}$$

Gradient of a vector

Operator

$$L = \nabla \underline{V} = \left(\underline{m}^j \frac{\partial}{\partial \xi_j}\right)\left(V^k \underline{m}_k\right)$$

$V^k \underline{m}_k = V^i \underline{m}_i$

$d\underline{m}_i = \Gamma_{ij}^{\;\;k}\, d\xi_j\, \underline{m}_k$

$$\frac{\partial V^k}{\partial \xi_j}\, \underline{m}_k \otimes \underline{m}^j + V^i\, \Gamma_{ij}^{\;\;k}\, \underline{m}_k \otimes \underline{m}^j$$

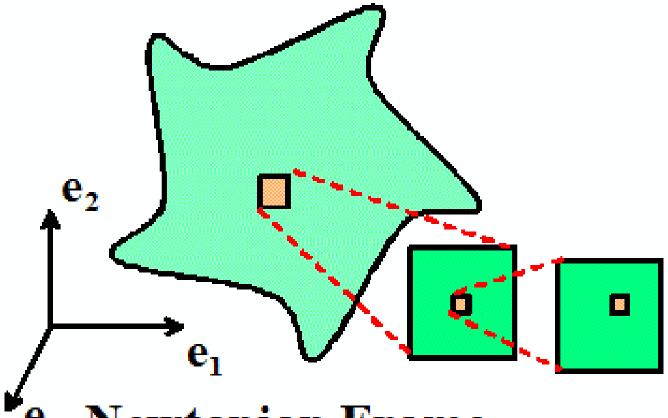
$$L^k_{\cdot j}\, \underline{m}_k \otimes \underline{m}^j = \left(\frac{\partial V^k}{\partial \xi_j} + V^i\, \Gamma_{ij}^{\;\;k}\right) \underline{m}_k \times \underline{m}^j$$

$$L^k_{\cdot j} = \frac{\partial V^k}{\partial \xi_j} + V^i\, \Gamma_{ij}^{\;\;k}$$

3 Kinematics of Deformable solids

3.1) Basic Assumptions

* A Newtonian frame exists
  - → Idealize as $\mathbb{R}^3$.
  - → Standard basis fixed



Newtonian Frame

* Matter is infinitely divisible
  Locally homogeneous