Review: Kinematics of Continua

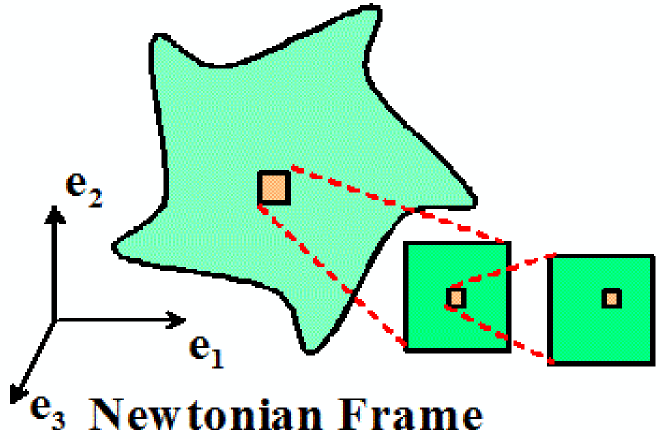
Assumptions:

1. Existence of a Newtonian Frame

   Idealized as $\mathbb{R}^3$

   Inertial basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$

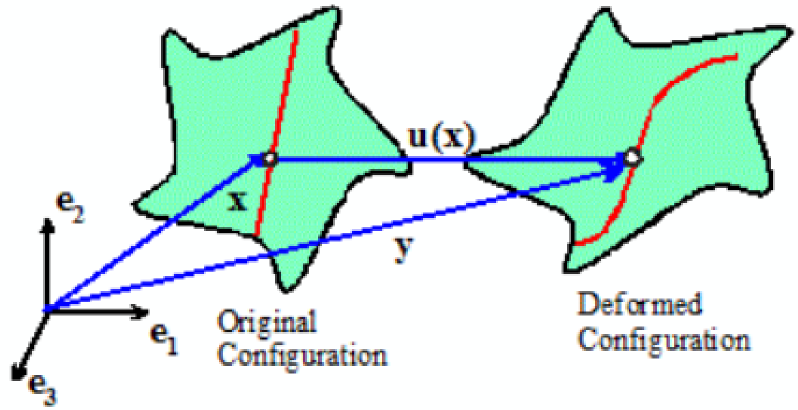
2. Matter is infinitely divisible, and locally homogeneous



3.2 Describing Deformation

"Configuration": Region of $\mathbb{R}^3$ occupied by solid or fluid

"Reference" config.: Any convenient fixed configuration – eg initial config



Current config: region occupied @ time t

$\underline{x}$: Position in reference config
$\underline{y}$: Position in current config

Deformation Mapping $\quad \underline{y}(\underline{x}, t)$: specifies position after deformation

$\underline{y}$ is continuous & differentiable

$$\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$$

* Displacement $\quad u_i = y_i - x_i$

* Particle velocity : $\quad v_i = \left.\frac{\partial y_i}{\partial t}\right|_{\underline{x}} = \left.\frac{\partial u_i}{\partial t}\right|_{\underline{x}}$

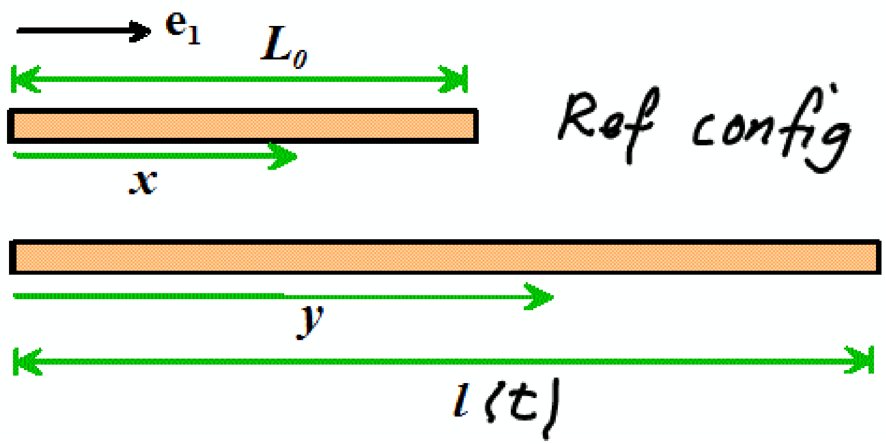
* Particle acceleration $\quad a_i = \left.\frac{\partial v_i}{\partial t}\right|_{\underline{x}} = \left.\frac{\partial^2 u_i}{\partial t^2}\right|_{\underline{x}}$

We can express pos, vel, accel as functions of $\underline{x}$ or $\underline{y}$

$$\underline{u}(\underline{x}),\quad \underline{v}(\underline{x}),\quad \underline{a}(\underline{x}) \quad \text{– Lagrangean}$$

$$\underline{u}^s(\underline{y}),\quad \underline{v}(\underline{y}),\quad \underline{a}(\underline{y}) \quad \text{– Eulerean}$$

Example: 1D stretch



Ref config

$$y = \frac{x}{L}\,l \Rightarrow u = x\left(\frac{l}{L} - 1\right)$$

$$v = \frac{x}{L}\frac{dl}{dt} \qquad a = \frac{x}{L}\frac{d^2 l}{dt^2}$$

$$u^s = y\left(1 - \frac{L}{l}\right)$$

Eiulerean

$$v^s = \frac{y}{l}\frac{dl}{dt} \qquad a^s = \frac{y}{l}\frac{d^2 l}{dt^2}$$

Eulerean displacement / vel / accel relations

$$V_i^s = \frac{\partial y_i^s}{\partial t}\Big|_y + \frac{\partial u_i^s}{\partial y_k} \underbrace{\frac{dy_k}{dt}}_{V_k^s} \quad \text{- particle vel}$$

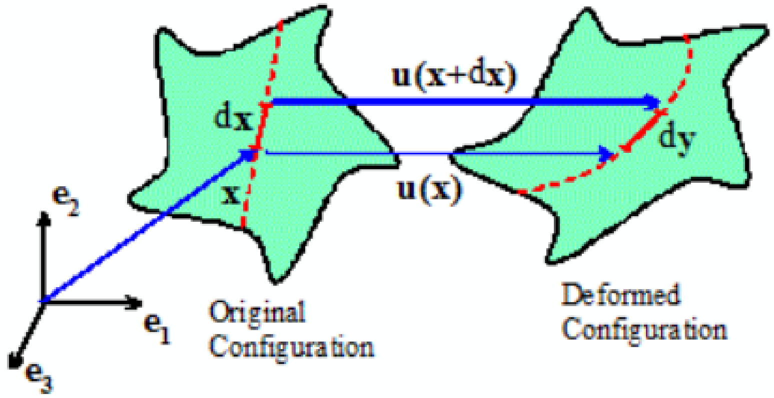
$$\left(\delta_{ik} - \frac{\partial u_i^s}{\partial y_k}\right) V_k^s = \frac{\partial u_i^s}{\partial t}\Big|_y$$

$$\left(\underline{I} - \underbrace{\nabla_y}_{\text{symbol for spatial gradient}} \underline{u}^s\right) \underline{v}^s = \frac{\partial \underline{u}^s}{\partial t}\Big|_y$$

$$a_i^s = \frac{\partial v_i^s}{\partial t}\Big|_y + \frac{\partial v_i}{\partial y_k} V_k^s$$

$$\underline{a}^s = \frac{\partial \underline{v}^s}{\partial t}\Big|_y + \left(\nabla_y \underline{V}^s\right) \underline{V}^s$$

Deformation Gradient



Define $F_{ij} = \dfrac{\partial y_i}{\partial x_j}$

$$F = \nabla \underline{y} = I + \nabla \underline{u}$$

Let $d\underline{x}$ be a material segment in ref config
 $d\underline{y}$ " " " " " current "

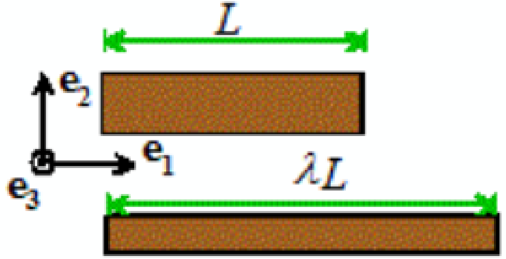
$$d\underline{y} = F\, d\underline{x} \qquad dy_i = F_{ij}\, dx_j$$

To see this note

$$d\underline{y} = \underline{y}(\underline{x} + d\underline{x}) - \underline{y}(\underline{x}) = \nabla \underline{y}\, d\underline{x} + O(|d\underline{x}|^2)$$

Volume preserving uniaxial extension

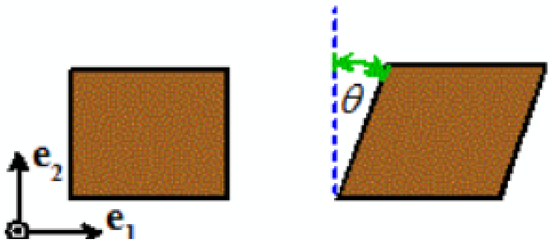
$$y_1 = \lambda x_1$$
$$y_2 = x_2 / \sqrt{\lambda}$$
$$y_3 = x_3 / \sqrt{\lambda}$$



$$F = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix}$$

Simple shear

$$y_1 = x_1 + \tan\theta\, x_2$$
$$y_2 = x_2$$
$$y_3 = x_3$$

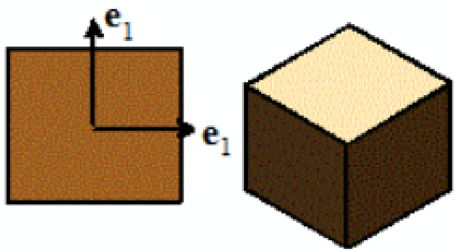


$$F = \begin{bmatrix} 1 & \tan\theta & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

General rigid rotation about the origin

$$\mathbf{y} = \mathbf{R} \cdot \mathbf{x} \quad \text{or} \quad y_i = R_{ij} x_j$$

where $\mathbf{R}$ must satisfy $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$, det($\mathbf{R}$)>0. (i.e. $\mathbf{R}$ is proper orthogonal). $\mathbf{I}$ is the identity tensor with components $\delta_{ik} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$



$F = R$

Use Rodriguez representation

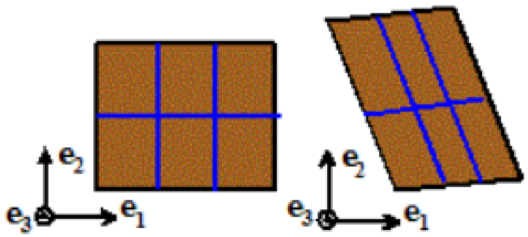
General homogeneous deformation

$$y_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + c_1$$
$$y_2 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + c_2$$
$$y_3 = A_{31}x_1 + A_{32}x_2 + A_3 x_3 + c_3$$

or

$$\mathbf{y} = \mathbf{A} \cdot \mathbf{x} + \mathbf{c} \quad y_i = A_{ij} x_j + c_i$$

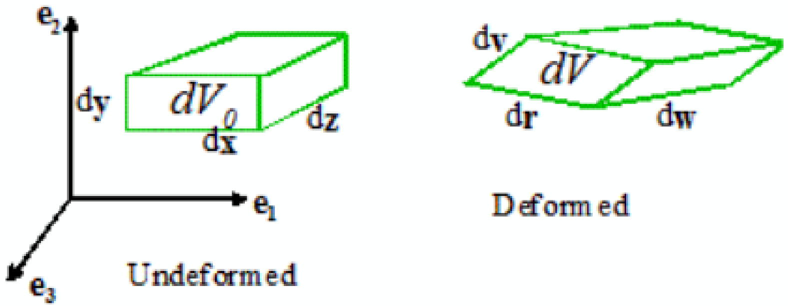
where A_{ij} are constants.



$F = A$

where A is independent of $\underline{x}$

Properties of F



Jacobian : $\boxed{J = \det(F)}$

Relates vol element dV_0 in ref config to dV in current config

$$\boxed{J = \frac{dV}{dV_0}}$$

Proof $dV_0 = d\underline{z}\cdot(d\underline{x}\times d\underline{y}) = dz_i\, \epsilon_{ijk}\, dx_j\, dy_k$

$$dV = d\underline{w}\cdot(d\underline{r}\times d\underline{v}) = F_{ip}\, dz_p\, \epsilon_{ijk}\, F_{jq}\, dx_q\, F_{kr}\, dy_r$$

$$= \epsilon_{pqr} \det(F)\, dz_p\, dx_q\, dy_r$$

$$= \det(F)\, dV_0$$