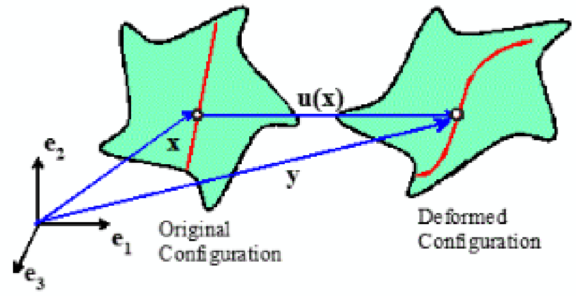
Review

Deformation Mapping

$$y_i = x_i + u_i(x_1, x_2, x_3, t)$$



Eulerian/Lagrangian descriptions of motion

$$y_i = x_i + u_i(x_j, t) \qquad \left.\frac{\partial y_i}{\partial t}\right|_{x_i=\text{const}} = \frac{\partial u_i}{\partial t} = v_i(x_j, t)$$

$$y_i = x_i + u_i(y_j, t) \qquad \left.\frac{\partial y_i}{\partial t}\right|_{x_i=\text{const}} = v_i(y_j, t) \qquad \left.\frac{\partial^2 y_i}{\partial t^2}\right|_{x_i=\text{const}} = a_i(y_j, t)$$

$$\left(\delta_{ik} - \frac{\partial u_i}{\partial y_k}\right)\left.\frac{\partial y_k}{\partial t}\right|_{x_i=\text{const}} = \left.\frac{\partial u_i}{\partial t}\right|_{y_i=\text{const}} \qquad \left.\frac{\partial^2 y_i}{\partial t^2}\right|_{x_i=\text{const}} = a_i(y_j, t) = \left.\frac{\partial v_i}{\partial t}\right|_{y_i=\text{const}} + v_k(y_j, t)\frac{\partial v_i}{\partial y_k}$$

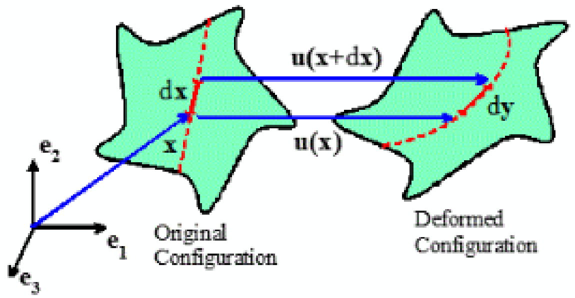
Deformation Gradient

$$\nabla \mathbf{y} = \nabla\left(\mathbf{x} + \mathbf{u}(\mathbf{x})\right) = \mathbf{F}$$

or $$\frac{\partial y_i}{\partial x_j} = \frac{\partial}{\partial x_j}\left(x_i + u_i\right) = \delta_{ij} + \frac{\partial u_i}{\partial x_j} = F_{ij}$$

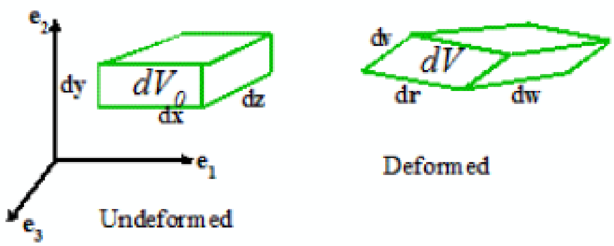
$$d\mathbf{y} = \mathbf{F} \cdot d\mathbf{x}$$

$$dy_i = F_{ik} dx_k$$



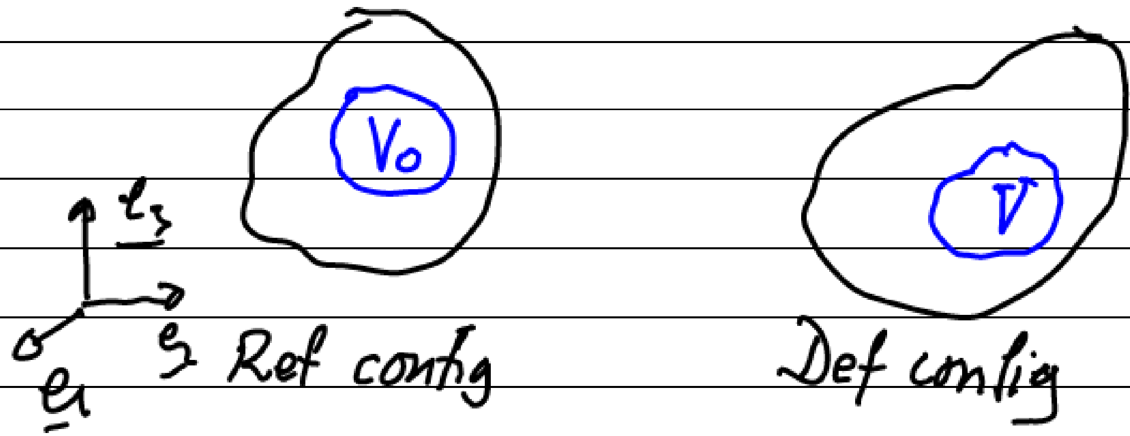
Jacobian

$$J = \det(\mathbf{F}) = \frac{dV}{dV_0}$$



Useful Observation

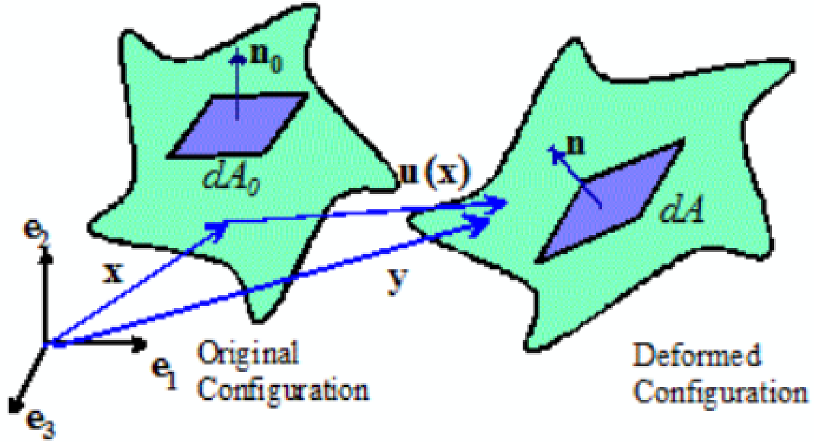
$$\int_V \phi \, dV = \int_{V_0} \phi \, J \, dV_0$$



ϕ - any scalar function of $\underline{x}$ or $\underline{y}$

Area Elements

$$\boxed{dA\,\underline{n} = J\,F^{-T}\,dA_0\,\underline{n}_0}$$



Proof: Let $d\underline{x}$, $d\underline{y}$ be two fibers passing through a pt

Let $d\underline{v} = F\,d\underline{x}$ $\quad d\underline{w} = F\,d\underline{y}$ $\qquad dA\,\underline{n} = d\underline{v} \times d\underline{w}$

$$dA\,n_i = \epsilon_{ijk}\,dv_j\,dw_k = \epsilon_{ijk}\,F_{jp}\,dx_p\,F_{kq}\,dy_q$$

$$= F^{-1}_{ri}\,\epsilon_{njk}\,F_{nr}\,F_{jp}\,F_{kq}\,dx_p\,dy_q \qquad F_{nr}F^{-1}_{ri} = \delta_{in}$$

$$= F^{-1}_{ri}\,\epsilon_{rpq}\,\det(F)\,dx_p\,dy_q$$

$A^T B = C_{ij}$

$A_{ki}\,B_{kj} = C_{ij}$

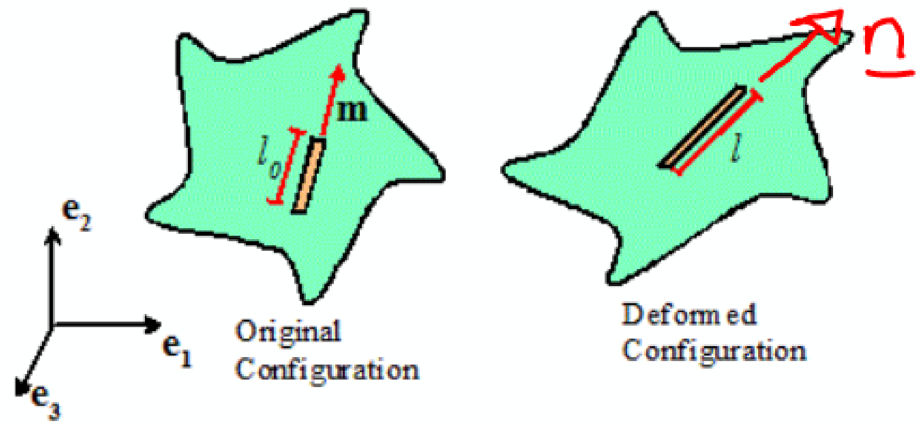
$$= J\,F^{-1}_{ri}\,\epsilon_{rpq}\,dx_p\,dy_q = J\,F^{-T}\,dA_0\,\underline{n}_0$$

Line Elements

Lagrange Strain Tensor

$$\boxed{E = \frac{1}{2}\left(F^T F - I\right)}$$

$$E_{ij} = \frac{1}{2}\left(F_{ki} F_{kj} - \delta_{ij}\right)$$



$$\frac{l^2 - l_0^2}{2 l_0^2} = m \cdot E m = m_i E_{ij} m_j$$

Proof: $\quad d\underline{y} = l \underline{n} \qquad d\underline{x} = l_0 \underline{m}$

$$l^2 - l_0^2 = d\underline{y} \cdot d\underline{y} - d\underline{x} \cdot d\underline{x} = (F d\underline{x}) \cdot F d\underline{x} - d\underline{x} \cdot d\underline{x}$$

$$= d\underline{x} \cdot \left(F^T F - I\right) d\underline{x}$$

$$l_0 m \cdot (2E)\, l_0 \underline{m} \qquad \text{QED}$$

Left and Right Cauchy-Green Strain tensors

* Right Cauchy-Green tensor $C = F^T F$
* Left " " " $B = F F^T$

$$\frac{\ell^2}{\ell_0^2} = m \cdot C\, \underline{m} \qquad \frac{\ell_0^2}{\ell^2} = \underline{n} \cdot B^{-1}\, \underline{n}$$

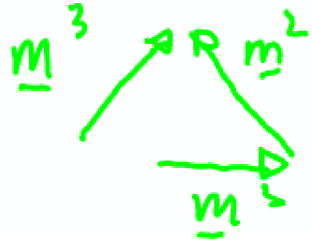
Eulerian strain tensor

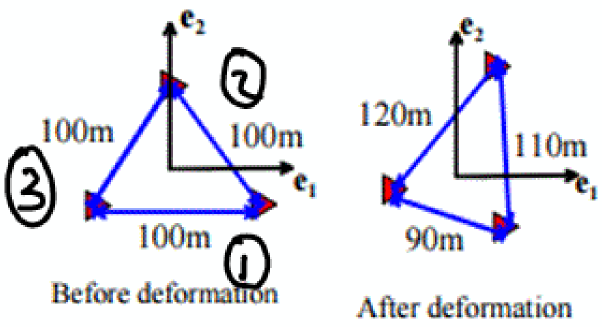
$$E^* = \frac{1}{2}\left(I - F^{-T} F^{-1}\right)$$

$$\frac{\ell^2 - \ell_0^2}{2\ell^2} = \underline{n} \cdot E^*\, \underline{n}$$

Example

Find the Lagrange Strain:





let $E = \begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}$

$\underline{m}^1 = (1 \quad 0)$

$\underline{m}^2 = (-1/2 \quad \sqrt{3}/2)$

$\underline{m}^3 = (1/2 \quad \sqrt{3}/2)$

$$\underline{m}^1 \cdot E\underline{m}^1 = (1 \quad 0)\begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = E_{11} = \frac{90^2 - 100^2}{2\ 100^2}$$

$$\underline{m}^2 \cdot E\underline{m}^2 = \left[-\tfrac{1}{2} \quad \sqrt{3}/2\right]\begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{11} \end{bmatrix}\begin{bmatrix} -\tfrac{1}{2} \\ \sqrt{3}/2 \end{bmatrix} = \frac{E_{11}}{4} + \frac{3}{4}E_{22} - \frac{\sqrt{3}}{2}E_{12} = \frac{110^2 - 100^2}{2\ 100^2}$$

$$\underline{m}^3 \cdot E\underline{m}^3 = \left[\tfrac{1}{2} \quad \sqrt{3}/2\right]\begin{bmatrix} E_{11} & E_{12} \\ E_{12} & E_{22} \end{bmatrix}\begin{bmatrix} \tfrac{1}{2} \\ \tfrac{\sqrt{3}}{2} \end{bmatrix} = \frac{E_{11}}{4} + \frac{3}{4}E_{22} + \frac{\sqrt{3}}{2}E_{12}$$

$$= \frac{120^2 - 100^2}{2\ 100^2}$$

$E_{11} = -\frac{19}{200}$ $\quad E_{22} = \frac{149}{600}$ $\quad E_{12} = \frac{23\sqrt{3}}{600}$

Polar Decomposition of F

Polar decomposition theorem : $F = RU = VR$

$U = U^T$ – Right stretch tensor

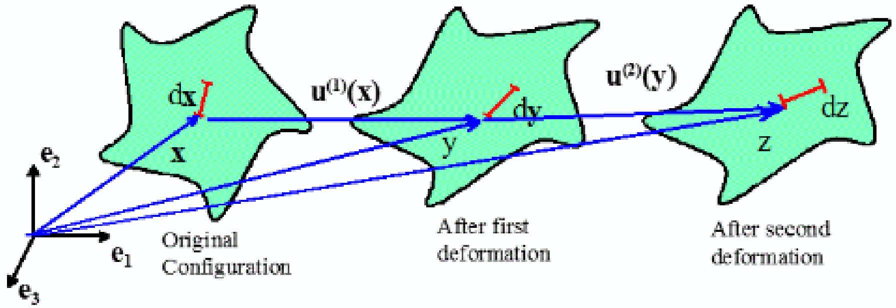
$V = V^T$ – Left " "

$RR^T = R^TR$ – Rotation tensor

Visualization

① $F = F^{(2)} F^{(1)}$ is a sequence of two deformations

$d\underline{z} = F^{(2)} dy = F^{(2)} F^{(1)} d\underline{x}$



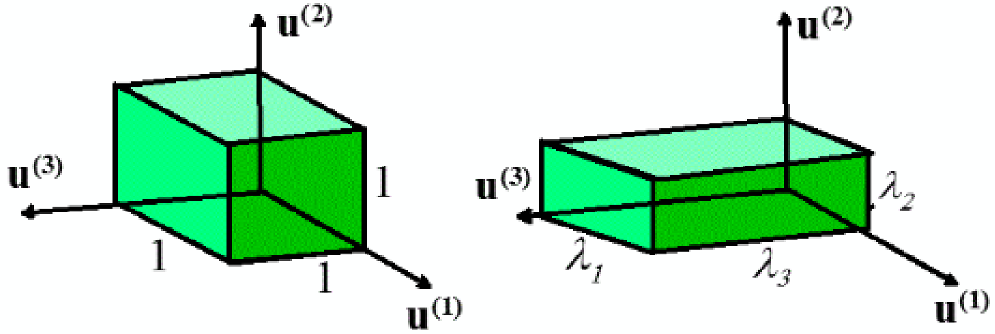
② R is a rigid rotation : preserves lengths and angles

③ U and V can be visualized using their eigenvalues & eigenvectors

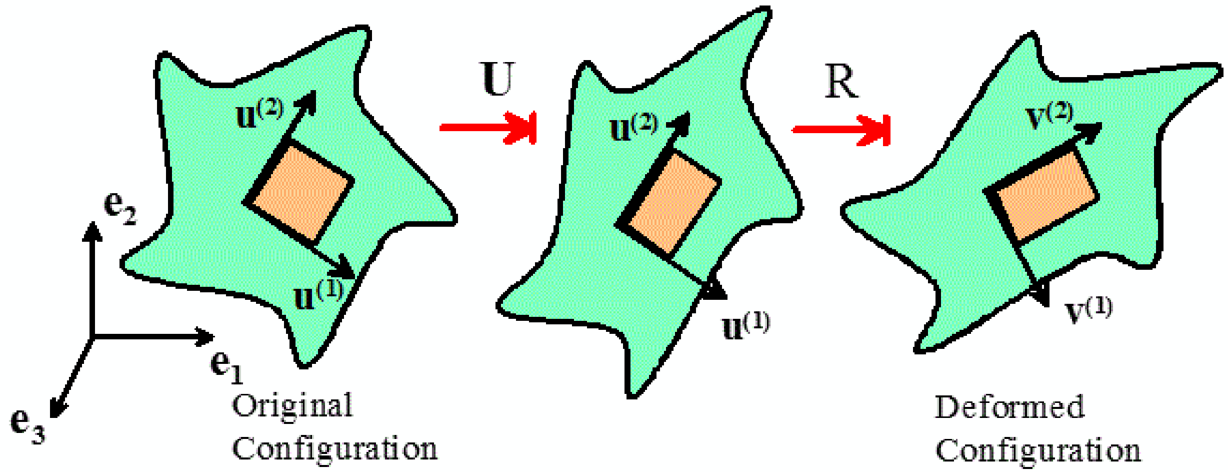
eg let $\{\lambda_i \; \underline{u}^{(i)}\}$ satisfy $\quad U\underline{u}^{(i)} = \lambda_i \underline{u}^{(i)}$

with $\underline{u}^{(i)} \cdot \underline{u}^{(j)} = \delta_{ij}$

$\Rightarrow$ material fibers parallel to $\underline{u}^{(i)}$ remain parallel to $\underline{u}^{(i)}$ but increase in length by a factor λ_i



Hence visualize $F = RU$ as below



Similarly: $F = VR$ is a rigid rotation followed by stretches parallel to principal axes of V

λ_i : Principal stretches

$\underline{u}^{(i)}, \underline{v}^{(i)}$ = principal stretch directions