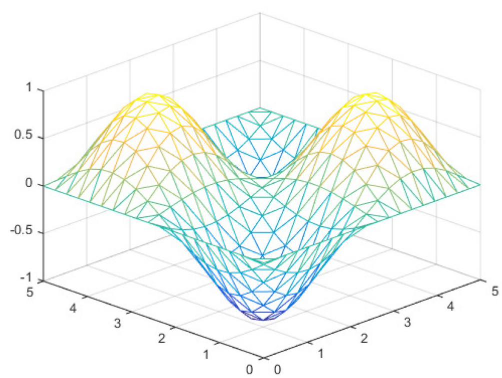
**School of Engineering**
**Brown University**

# EN2340: Computational Methods in Structural Mechanics

## Homework 2: Basic FEA procedures

The main goal of the homework is to get comfortable with the book-keeping operations needed perform the various FEA tasks.

You can use the constant strain linear elastic finite element MATLAB code posted on the Notes page of the class website as a general template for your calculations. The biggest differences between the elasticity problem discussed in class and the problems you will solve here are (1) you only need to calculate values for a single degree of freedom, instead of the displacement vector. This makes the calculation a lot simpler, but you will have to work through how to modify the book-keeping in the MATLAB code to change the number of degrees of freedom; and (2) You need to solve an eigenvalue problem instead of solving a system of linear equations for the solution. MATLAB will do this for you with just one line of code.




In solid mechanics, the two-dimensional Helmholtz equation governs the vibration modes and natural frequencies of a vibrating membrane. Specifically, consider a flat, taught membrane, with fixed edge, which occupies some two-dimensional area $A$ in the $(x_1, x_2)$ plane. Let $u(x_1, x_2)$ denote the transverse displacement of the membrane. The displacement for the $n$th vibration mode satisfies

$$\frac{\partial^2 u}{\partial x_i \partial x_i} + k_n^2 u = 0$$

where $k_n$ is the wave-number (related to the wave speed $c$ and natural frequency $\omega_n$ by $k_n = \omega_n / c$ ). This equation only has non-zero real-valued solutions for special values of $k_n$, which depend on the shape of the membrane.

It can be shown (many of you probably know how…) that the displacements satisfying this equation are stationary points (not a minimum) [1] of the functional

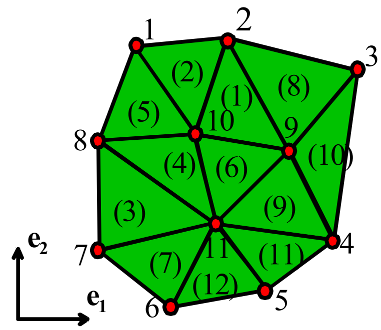
$$\Pi = \int_A \frac{1}{2}\left(\nabla u \cdot \nabla u - k_n^2 u^2\right) dA$$

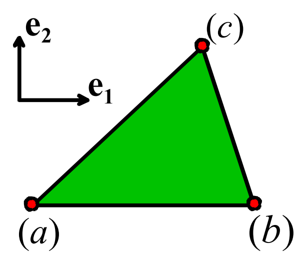
[1] For some recent literature on variational principles for the Helmholtz equation see
https://www.reading.ac.uk/web/FILES/maths/preprint_12_23_Moiola.pdf
http://people.bath.ac.uk/eas25/ibps.pdf
http://content.lib.utah.edu/utils/getfile/collection/etd2/id/1465/filename/739.pdf

In this problem you will develop a two-dimensional FEA code in MATLAB to solve this equation. The overall approach will be to use the variational statement of the equation to derive a system of linear equations of the form

$$[K]\underline{U} = k_n^2 [M]\underline{U}$$

for the displacements at a set of discrete points inside the membrane. This is generalized eigenvalue problem – the eigenvalues $k_n$ and eigenvectors $\underline{u}$ specify the wave numbers and mode shapes for the vibration modes.




To implement this procedure, the basic scheme for linear elasticity discussed in class (see also chapter 7 of http://solidmechanics.org ) will be modified as follows:

1. We identify a set of 'nodes' in the solid where we want to calculate $u$
2. Divide the solid into triangles with the nodes at the corners
3. Interpolate the transverse displacement $u$ in each triangle linearly between values at the nodes. So, for a generic triangle with corners $a,b,c$

$$u = N^a u^a + N^b u^b + N^c u^c$$

where the interpolation functions are exactly the same as those used for the elasticity problem.

4. We now need to calculate the contribution to $\Pi$ from each element. We can compute the contribution to $\Pi$ from just one element as

$$\pi^{elem} = \frac{1}{2}\left(\underline{u}^{elem\,T} k^{elem}\ \underline{u}^{elem} - k_n^2 \underline{u}^{elem\,T}\ m^{elem}\ \ \underline{u}^{elem}\right)$$

$$k^{elem} = \int_A [B]^T [B] dA \qquad m^{elem} = \int_A \underline{N} \otimes \underline{N} dA$$

$$\underline{u}^{elem} = \begin{bmatrix} u^a \\ u^b \\ u^c \end{bmatrix} \qquad [B] = \begin{bmatrix} \partial N^a / \partial x_1 & \partial N^b / \partial x_1 & \partial N^c / \partial x_1 \\ \partial N^a / \partial x_2 & \partial N^b / \partial x_2 & \partial N^c / \partial x_2 \end{bmatrix} \qquad \underline{N} = \begin{bmatrix} N^a & N^b & N^c \end{bmatrix}$$

where $\otimes$ denotes an outer product (i.e. the matrix with components $m_{ab} = N^a N^b$ )

We need a painless way to evaluate the area integrals. Since $u$ varies linearly inside the triangle $\nabla u$ must be constant. Therefore

$$k^{elem} = \int_A [B]^T [B] dA \ = A_{elem} [B]^T [B]$$

A simple way to evaluate $m^{elem}$ is to compute the integrand at the center of each of the element edges and sum them, as follows:

$$m^{elem} = \int_A \underline{N} \otimes \underline{N} dA = \frac{A_{elem}}{3} \sum_{edges} \underline{N}(x_{center}, y_{center}) \otimes \underline{N}(x_{center}, y_{center})$$

5. Add up the contributions from each element to express the total potential $\Pi$ in terms of the displacements at each node, where

$$\Pi = \frac{1}{2}\left(\underline{U}^T [K] \underline{U} - k_n^2 \underline{U}^T [M] \underline{U}\right) \qquad [K] = \sum_{elements} k^{elem} \qquad [M] = \sum_{elements} m^{elem}$$

where $\underline{U}$ is a big vector consisting of the unknown transverse displacements at each node in the entire mesh.

6. Make the functional stationary – follow the procedure discussed in class for linear elasticity - this gives a linear eigenvalue problem for $\underline{U}$
7. Solve the eigenvalue problem, and post-process the results to display them.

Your mission is to implement this procedure as a MATLAB code, which should contain the following sections:

1. Generate the arrays defining the mesh and boundary conditions. It will be easiest to generate them directly using MATLAB rather than reading them from a file. You can make the membrane any shape you like, but a rectangular membrane is the easiest. Note that MATLAB has a function that will generate the connectivity for a mesh of triangular elements connecting a set of points automatically:
```
connect = delaunay(x,y)
```
Here, x and y are 1D vectors containing the x,y coordinates of the points. Note that connect(lmn,a) gives the number of the point corresponding to the ath node on element number lmn (this is the same convention used in the simple sample MATLAB constant strain triangle code)
2. Assemble the global $K$ and $M$ matrices. This will require:
    a. Loop over all the elements
    b. For the current element, compute $k^{element}$ and $m^{element}$ (it is cleanest to do this by creating a function that calculates the element matrices given coordinates of the corners. Note that the coordinates of the center of a triangle is just the average of the coordinates of its corners)
    c. Add the element matrices into the right rows and columns of the global stiffness matrix
3. Modify the equations to enforce zero accelerations at nodes on the boundary. This means
    a. Loop over the nodes at the edge of the membrane
    b. Find the row $K$ corresponding to the node, and set it to zero. Find the row in $M$ corresponding to the node, and set it to zero, with a 1 on the diagonal.
4. Now solve the equations: in MATLAB you can use
```
[V,D] = eig(K,M);
```
to find the eigenvectors (each column of V is an eigenvector; D is a diagonal matrix with eigenvalues). You will find that the system has a large number of zero eigenvalues – we are only interested in the mode shapes for nonzero eigenvalues.
5. For plotting purposes you will want to sort the eigenvalues by size and then plot the mode shapes. The following code will find the lowest non-zero eigenvalue and plot the corresponding mode shape. You can also look at the higher modes by adding integers to i after the loop.
```
[kk,index] = sort(diag(D));
for i = 1:length(kk)
    if (kk(i)>0) break; end
end
kk(i)
trimesh(connect,coord(:,1),coord(:,2),V(:,index(i)));
```

The exact solution for the natural frequencies and mode-shapes of a rectangular membrane occupying $0<x<a$ , $0<y<b$ is

$$k_{nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \qquad u_{nm} = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

You can compare your predictions with the exact solution (you will find that convergence is rather slow).

As a solution to this assignment, please submit

1. Your MATLAB code (please upload to canvas)
2. A 1-page description of some sample calculations, including:
    a. Plots of a few mode shapes for a membrane with geometry of your choice (give the wave number for each mode)
    b. A test that compares the exact solution and numerically predicted wave numbers for a membrane (a rectangular membrane is fine, but you can do other shapes too) for several different mesh densities.