EN234: Computational methods in Structural and Solid Mechanics

Homework 5: Advanced elements – small strain incompatible mode element
Due Fri Oct 20, 2017

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In this homework you will implement a small-strain, 2D (plane strain) version of an 'incompatible mode' element to improve the accuracy of the element.

The basic idea was discussed in class (see the notes online). You can also find a (somewhat inefficient) MATLAB example on the notes page of the website. Your goal is to implement this procedure as an ABAQUS UEL (you can simply modify the UEL you wrote in Homework 3).

The stiffness matrix and residual vector for an incompatible mode element are computed as follows:

1. Add internal degrees of freedom α_i^n with $n=1..2$ and DOF $i=1..2$ to the element
2. Define an augmented **B** matrix that maps element DOF to element strains as follows

$$
\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \end{bmatrix} =
\begin{bmatrix}
\dfrac{\partial N^1}{\partial x_1} & 0 & \dfrac{\partial N^2}{\partial x_1} & 0 & \dfrac{\eta(0)}{\eta(\xi)}\xi_1\dfrac{\partial \xi_1}{\partial x_1} & 0 & \dfrac{\eta(0)}{\eta(\xi)}\xi_2\dfrac{\partial \xi_2}{\partial x_1} & 0 \\
0 & \dfrac{\partial N^1}{\partial x_2} & 0 & \dfrac{\partial N^2}{\partial x_2} \cdots & 0 & \dfrac{\eta(0)}{\eta(\xi)}\xi_1\dfrac{\partial \xi_1}{\partial x_2} & 0 & \dfrac{\eta(0)}{\eta(\xi)}\xi_2\dfrac{\partial \xi_2}{\partial x_2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\dfrac{\partial N^1}{\partial x_2} & \dfrac{\partial N^1}{\partial x_1} & \dfrac{\partial N^2}{\partial x_2} & \dfrac{\partial N^2}{\partial x_1} & \dfrac{\eta(0)}{\eta(\xi)}\xi_1\dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\eta(0)}{\eta(\xi)}\xi_1\dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\eta(0)}{\eta(\xi)}\xi_2\dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\eta(0)}{\eta(\xi)}\xi_2\dfrac{\partial \xi_2}{\partial x_1}
\end{bmatrix}
\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ \vdots \\ \alpha_1^1 \\ \alpha_2^1 \\ \alpha_1^2 \\ \alpha_2^2 \end{bmatrix}
$$

Here $-1 < \xi_i < 1$ $i = 1,2$ are local element coordinates, $x_i = N^a(\xi)x_i^a$ are physical coordinates, N^a are the standard element interpolation functions, and $\eta = \det(dx_i / d\xi_j)$.

3. Compute the intermediate element stiffness and internal force vector using the usual expressions

$$\hat{\mathbf{r}} = - \int_{V_e^{(l)}} \mathbf{B}^T \boldsymbol{\sigma} dV \qquad \hat{\mathbf{k}} = \int_{V_e^{(l)}} \mathbf{B}^T \mathbf{D} \mathbf{B} dV$$

where **D** is the usual matrix of elastic constants for plane strain deformation and $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

4. Extract the sub-matrices $\mathbf{k}^{uu}, \mathbf{k}^{\alpha\alpha}, \mathbf{k}^{u\alpha}, \mathbf{k}^{\alpha u}$ and the vectors $\mathbf{r}^u, \mathbf{r}^\alpha$ from $\hat{\mathbf{k}}, \hat{\mathbf{r}}$

$$\hat{\mathbf{k}} = \begin{bmatrix} \mathbf{k}^{uu} & \mathbf{k}^{u\alpha} \\ \mathbf{k}^{\alpha u} & \mathbf{k}^{\alpha\alpha} \end{bmatrix} \qquad \hat{\mathbf{r}} = \begin{bmatrix} \mathbf{r}^u \\ \mathbf{r}^\alpha \end{bmatrix}$$

Here, $\mathbf{k}^{\alpha\alpha}$ is a 4x4 matrix for a 2D element.

5. Eliminate the internal degrees of freedom α_i^n to assemble the element stiffness and internal force vectors:

$$\mathbf{k} = \mathbf{k}^{uu} - \mathbf{k}^{u\alpha}\mathbf{k}^{\alpha\alpha-1}\mathbf{k}^{\alpha u} \qquad \mathbf{r} = \mathbf{r}^{u} - \mathbf{k}^{u\alpha}\mathbf{k}^{\alpha\alpha-1}\mathbf{r}^{\alpha}$$

The calculations have to be done in the right order, because you need to know the values of the internal degrees of freedom $\boldsymbol{\alpha}$ to be able to calculate stresses in the element. The following sequence works:

1. Calculate $\hat{\mathbf{k}}$ (assemble contributions from each integration point in the element in the usual way
2. Calculate $\mathbf{k}^{uu}, \mathbf{k}^{\alpha\alpha}, \mathbf{k}^{u\alpha}, \mathbf{k}^{\alpha u}$,
3. Calculate the internal degrees of freedom $\boldsymbol{\alpha}$ from the element displacements $\mathbf{u}$ using
   $$\boldsymbol{\alpha} = -\mathbf{k}^{\alpha\alpha-1}(\mathbf{k}^{\alpha u}\mathbf{u})$$
4. Calculate $\hat{\mathbf{r}}$ (this will need a separate loop over the integration points). The strains can be calculated using $\mathbf{B}$ and $\mathbf{u}, \boldsymbol{\alpha}$, and the stresses can be calculated from the stress-strain relation
5. Extract. $\mathbf{r}^{u}, \mathbf{r}^{\alpha}$
6. Finally, assemble the element stiffness and residual $\mathbf{k} = \mathbf{k}^{uu} - \mathbf{k}^{u\alpha}\mathbf{k}^{\alpha\alpha-1}\mathbf{k}^{\alpha u} \quad \mathbf{r} = \mathbf{r}^{u} - \mathbf{k}^{u\alpha}\mathbf{k}^{\alpha\alpha-1}\mathbf{r}^{\alpha}$

Please implement this procedure by extending the 2D linear elastic element that you coded in Homework 3. You should write your code so that it can act both as a conventional 2D linear elastic element, as well an incompatible mode element, according to the value of the variable JTYPE in the ABAQUS UEL. Note that the template provided with HW3 already defined vectors and matrices to store $\mathbf{k}^{uu}, \mathbf{k}^{\alpha\alpha}, \mathbf{k}^{u\alpha}, \mathbf{k}^{\alpha u}$, $\mathbf{r}^{u}, \mathbf{r}^{\alpha}$

To test your code in EN234FEA, you can find a sample input file called Abaqus_uel_cantilever.in in the input_files directory of EN234FEA. It identifies the user element type as U2, which will set JTYPE=2 inside the UEL (you could use U1, JTYPE=1 to indicate a conventional element). The input file calculates deformation and stress in a cantilever beam subjected to a concentrated force on its end.

You can test your element in ABAQUS as well if you are curious (you will need to create your own .inp file) but this is not required.

As a solution to this problem, please
   (1) Upload a 1 or 2 page pdf that shows your plots the deformation and stress (as contours) in the cantilever beam, using both conventional and incompatible mode elements, with some discussion of the results (eg how you decided that your solution is correct).
   (2) Push your code to Github.