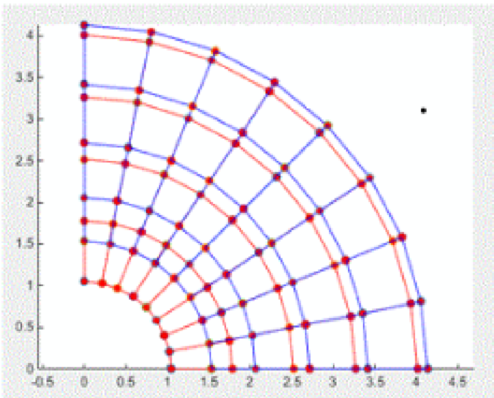
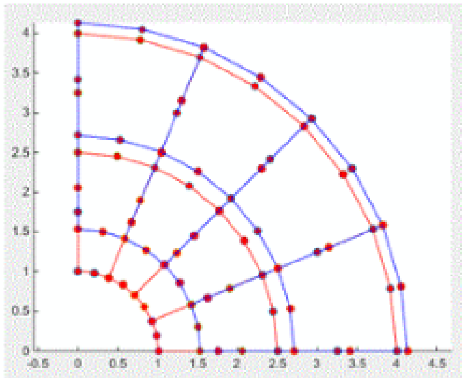
Review

- Volumetric Locking
  - For near-incompressible materials, all conventional elements ‘lock’ and predict zero displacement solutions for most loading
  - Problem is caused by too many incompressibility constraints (one for each integration point) compared to number of DOFs



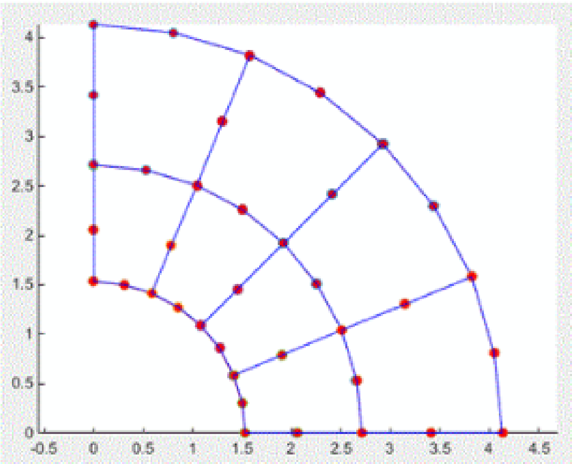


Review

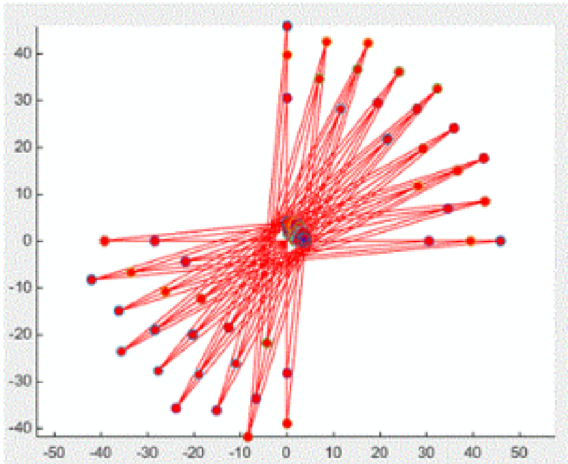
Fix #1 (works for some elements) – reduced integration

- Use one order lower integration scheme

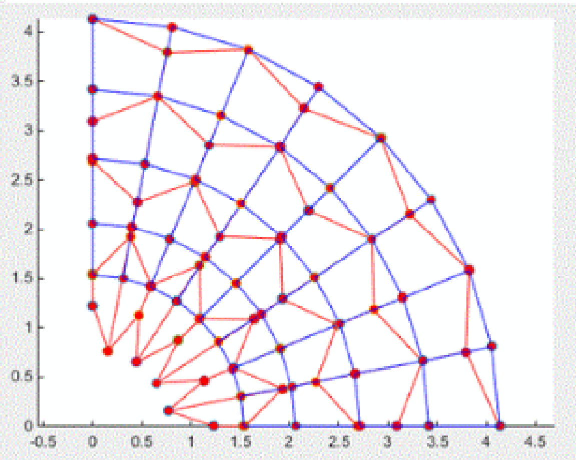
Number of integration points for reduced integration schemes	
Linear triangle (3 nodes) 1 point	Linear tetrahedron (4 nodes): 1 point
Quadratic triangle (6 nodes): 3 points	Quadratic tetrahedron (10 nodes): 4 points
Linear quadrilateral (4 nodes): 1 point	Linear brick (8 nodes): 1 point
Quadratic quadrilateral (8 nodes): 4 points	Quadratic brick (20 nodes): 8 points



8 noded quads – perfect!



4 noded quads – Hourglassing



Scaled to show hourglass mode

Topics for todays class

Fixes for locking

- R.I. with hourglass control (works for all element types; choice and design of hourglass stiffness is tricky)
- Selective reduced integration (works for all elements; hard to extend to finite strains)
- B-bar method (works for all elements; finite strain version exists)

Fix #2 Reduced Integration with hourglass control

Basic Idea: Add artificial stiffness to resist hourglassing

Approach: Introduce "Hourglass base vectors"

Hourglass base vectors	
Linear quadrilateral	$\Gamma^{a(1)} = (+1, -1, +1, -1)$
Linear brick	$\Gamma^{a(1)} = (+1, +1, -1, -1, -1, -1, +1, +1)$ $\Gamma^{a(2)} = (+1, -1, -1, +1, -1, +1, +1, -1)$ $\Gamma^{a(3)} = (+1, -1, +1, -1, +1, -1, +1, -1)$ $\Gamma^{a(4)} = (-1, +1, -1, +1, +1, -1, +1, -1)$

These describe hourglassing in square / cubic elements

Correction for arbitrary geometry

$$\gamma^{a(m)} = \Gamma^{a(m)} - \sum_{b=1}^{\#\,nodes} \Gamma^{b(m)} x_j^b \frac{\partial N^a}{\partial x_j}(0)$$

Now introduce modified el stiffness

$$k_{aibk} = \int_{\Omega_{el}} C_{ijkl} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial x_j} dV + \beta \Omega_{el} \sum_{m=1}^{\#\text{ modes}} \gamma^{a(m)} \gamma^{b(m)}$$

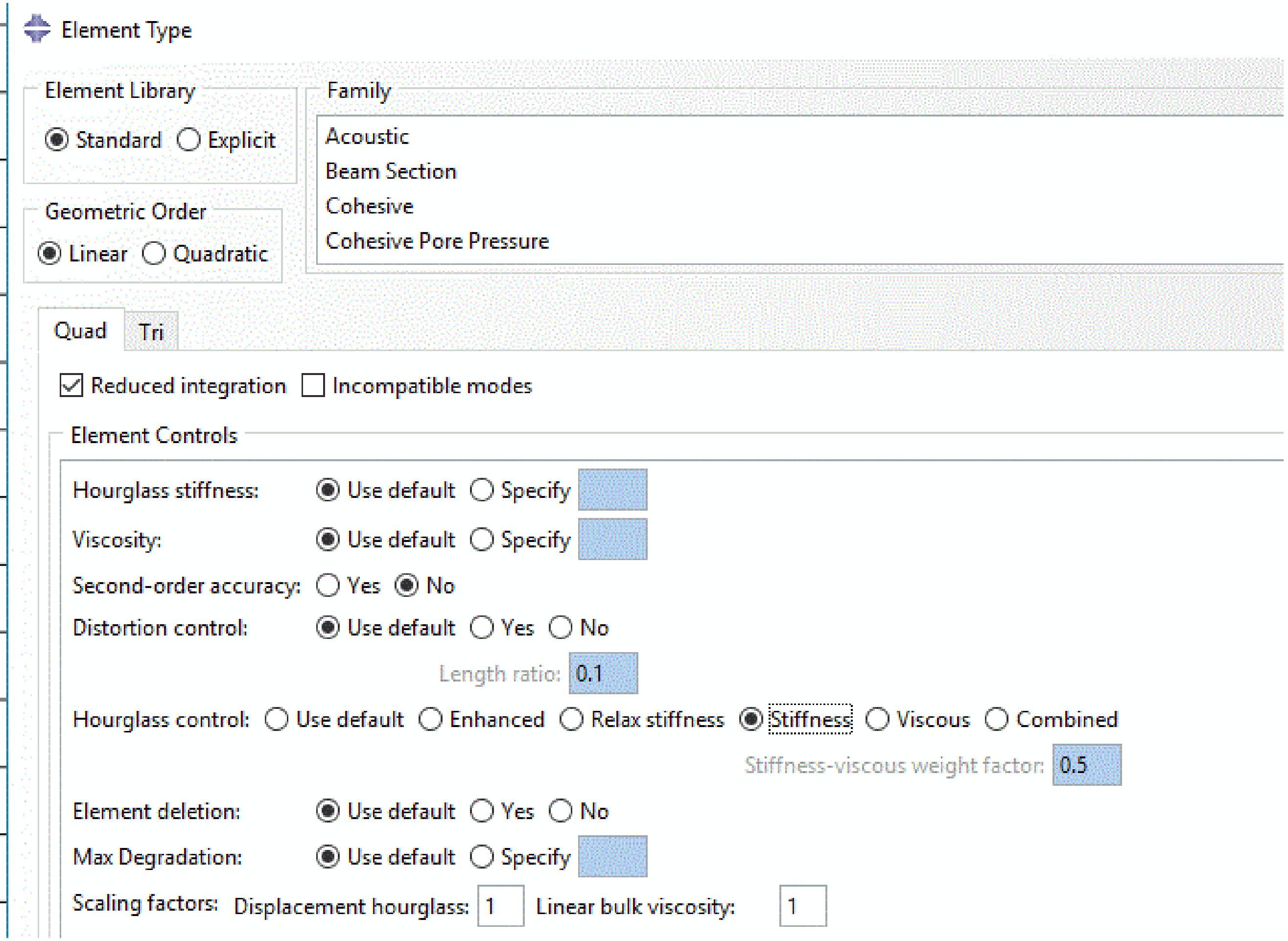
Ω_{el} - el volume

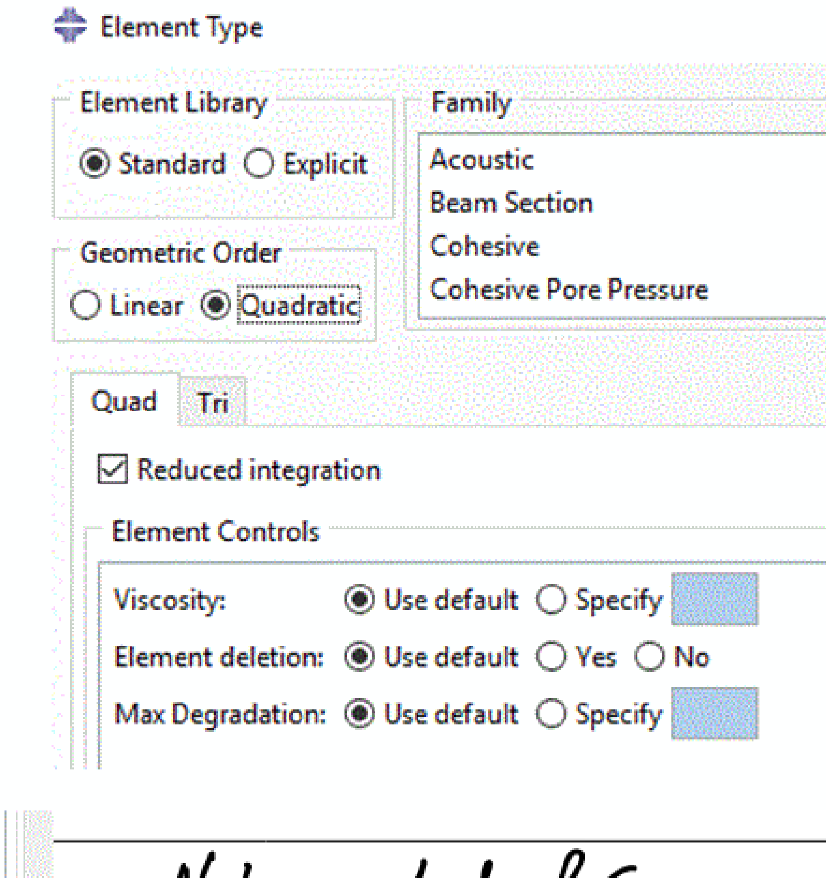
β - "hourglass stiffness" typically ~ 0.05 x shear modulus

Implementation : 2D

$$[k_{el}] = \sum_{\substack{\text{reduced} \\ \text{int pts}}} [B]^T [D] [B] \eta w_i + \beta \Omega_{el} \begin{bmatrix} \gamma^1 \gamma^1 & \gamma^1 \gamma^1 & \gamma^1 \gamma^2 \\ \gamma^1 \gamma^1 & & \\ \gamma^2 \gamma^1 & & \\ \vdots & & \end{bmatrix}$$

ABAQUS menus:

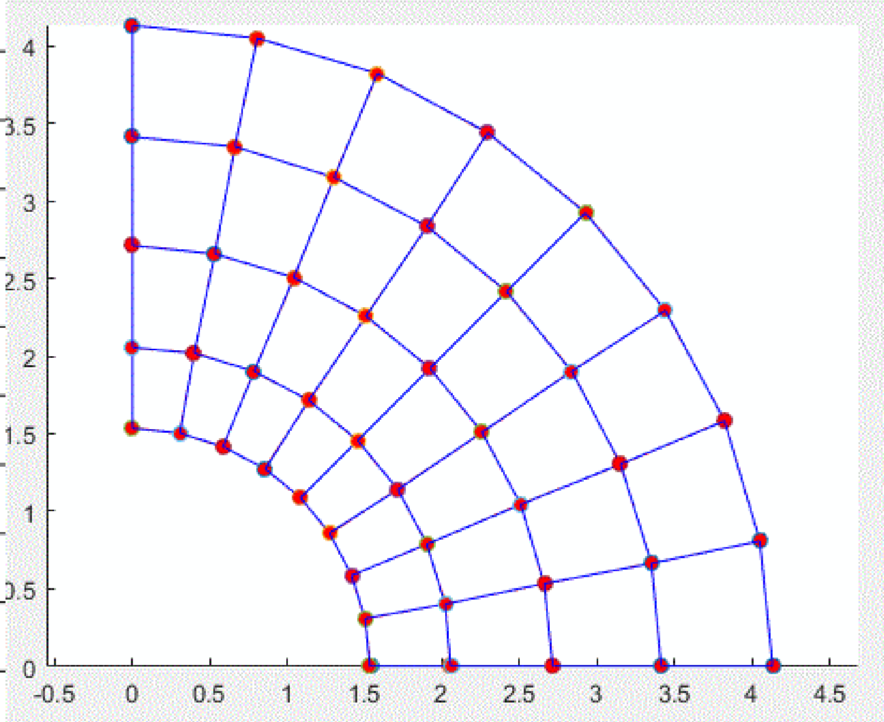




Not needed for quadratic elements

This stabilizes all elements:

- Advantage: Speed
  - good for explicit
- Disadvantages: choice of β can be tricky



Stabilization can fail for large deformations

Fix #3: "Selective reduced integration"

- Basic idea: separate volumetric & shear modes in element and integrate separately
- Approach: Define "deviatoric" strain

$$e_{ij} = \varepsilon_{ij} - \varepsilon_v \delta_{ij} \quad \Leftarrow \text{Shear}$$

$$\left.\begin{aligned} \varepsilon_v &= \varepsilon_{kk}/3 \quad (3D) \\ \varepsilon_v &= (\varepsilon_{11}+\varepsilon_{22})/2 \quad (2D) \end{aligned}\right\} \text{Volumetric Strains}$$

Define new [B] matrices

$$\underline{e} = [B^{DEV}]\,\underline{u} \qquad \underline{\varepsilon}^{vol} = [B^{vol}]\,\underline{u}$$

$\underline{e}$, $\underline{\varepsilon}^{vol}$ stored as before

2D [B] matrices

$$\underline{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ 2e_{12} \end{bmatrix} \qquad \underline{\varepsilon^{VOL}} = \frac{1}{2}\begin{bmatrix} \varepsilon_{11}+\varepsilon_{22} \\ \varepsilon_{11}+\varepsilon_{22} \\ 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial N^1}{\partial x_1} & 0 & \frac{\partial N^2}{\partial x_1} & 0 & \frac{\partial N^3}{\partial x_1} & 0 \\ 0 & \frac{\partial N^1}{\partial x_2} & 0 & \frac{\partial N^2}{\partial x_2} & 0 & \frac{\partial N^3}{\partial x_2} \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^3}{\partial x_2} & \frac{\partial N^3}{\partial x_1} \end{bmatrix}$$

$$[B^{VOL}] = \frac{1}{2}\begin{bmatrix} \frac{\partial N^1}{\partial x_1} & \frac{\partial N^1}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^3}{\partial x_1} & \frac{\partial N^3}{\partial x_2} \\ \frac{\partial N^1}{\partial x_1} & \frac{\partial N^1}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^3}{\partial x_1} & \frac{\partial N^3}{\partial x_2} \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B^{DEV}] = [B] - [B^{VOL}]$$

3D [B] matrices

$$\underline{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \\ e_{13} \\ e_{23} \end{bmatrix} \qquad \underline{\varepsilon^{VOL}} = \frac{1}{3}\begin{bmatrix} \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} \\ \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} \\ \varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} \frac{\partial N^1}{\partial x_1} & 0 & 0 & \frac{\partial N^2}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial N^1}{\partial x_2} & 0 & 0 & \frac{\partial N^2}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial N^1}{\partial x_3} & 0 & 0 & \frac{\partial N^2}{\partial x_3} \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_1} & 0 & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & 0 \\ \frac{\partial N^1}{\partial x_3} & 0 & \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_3} & 0 & \frac{\partial N^2}{\partial x_1} \\ 0 & \frac{\partial N^1}{\partial x_3} & \frac{\partial N^1}{\partial x_2} & 0 & \frac{\partial N^2}{\partial x_3} & \frac{\partial N^2}{\partial x_2} \end{bmatrix} \cdots\cdots$$

$$[B^{VOL}] = \frac{1}{3}\begin{bmatrix} \frac{\partial N^1}{\partial x_1} & \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_3} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_3} \\ \frac{\partial N^1}{\partial x_1} & \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_3} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_3} \\ \frac{\partial N^1}{\partial x_1} & \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_3} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_3} \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[B^{DEV}] = [B] - [B^{VOL}]$$

Now assemble $[k^{el}]$ with new $[B]$ matrices

$$[k^{el}] = \int_{\Omega_e} [B^{DEV} + B^{VOl}]^T [D] [B^{DEV} + B^{VOl}]\, dV$$

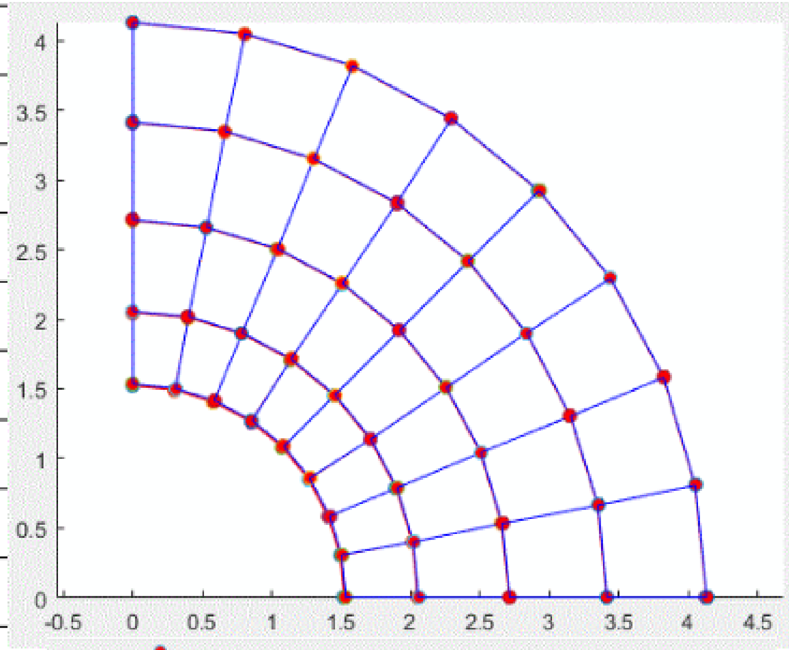
$$\underline{R} = \int_{\Omega_0} [\overbrace{B^{DEV}}^{\text{Full}} + \underbrace{B^{VOl}}_{\text{Reduced}}]\, \underline{\sigma}^0\, dV$$

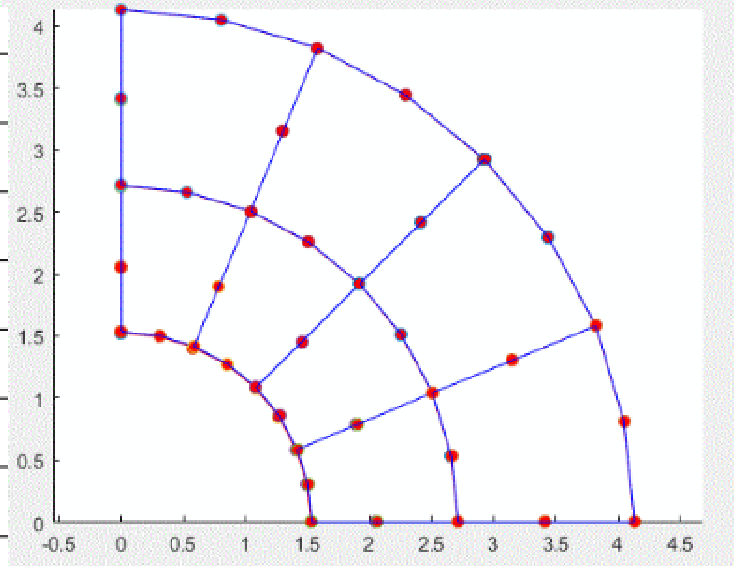
Note that $[B^{DEV}]^T [D] [B^{VOl}] = 0$

$$\Rightarrow [k^{el}] = \underbrace{\int_{\Omega_{el}} [B^{DEV}]^T [D] [B^{DEV}]\, dV}_{\text{Evaluate with full integration}} + \underbrace{\int_{\Omega_{el}} [B^{VOl}]^T [D] [B^{VOl}]\, dV}_{\text{Reduced int}}$$

This cures locking in all element types

- Easy to implement
- Hard to extend to finite strains





Fix #4 "B-bar" method

Alternative approach to selective reduced integration

- Introduce a new strain field inside each element

Let $\varepsilon_{ij} = e_{ij} + \varepsilon_v \delta_{ij}$

We want ε_v to be constant throughout element

Introduce vol averaged volumetric strain

$$\omega = \frac{1}{\Omega_{el}} \int_{\Omega_{el}} \varepsilon_v \, dV$$

Let $\bar{\varepsilon}_{ij} = e_{ij} + \omega \delta_{ij}$

Implementation :

Define new B matrix : $\bar{\varepsilon} = [\bar{B}]\,\underline{u}$

Can express $\bar{B}$ in terms of vol averaged shape function derivs

$$\overline{\frac{\partial N^a}{\partial x_j}} = \frac{1}{\Omega_{el}} \int_{\Omega_{el}} \frac{\partial N^a}{\partial x_j}\, dV$$

Then 2D $\bar{B}$ matrix is :

$$[\bar{B}] = \begin{bmatrix} \frac{\partial N^1}{\partial x_1} & 0 & \frac{\partial N^2}{\partial x_1} & 0 & \frac{\partial N^3}{\partial x_1} & 0 \\ 0 & \frac{\partial N^1}{\partial x_2} & 0 & \frac{\partial N^2}{\partial x_2} & 0 & \frac{\partial N^3}{\partial x_2} \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^3}{\partial x_2} & \frac{\partial N^3}{\partial x_1} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \frac{\partial N^1}{\partial x_1} & \frac{\partial N^1}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^3}{\partial x_1} & \frac{\partial N^3}{\partial x_2} \\ \frac{\partial N^1}{\partial x_1} & \frac{\partial N^1}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^3}{\partial x_1} & \frac{\partial N^3}{\partial x_2} \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \overline{\frac{\partial N^1}{\partial x_1}} & \overline{\frac{\partial N^1}{\partial x_2}} & \overline{\frac{\partial N^2}{\partial x_1}} & \overline{\frac{\partial N^2}{\partial x_2}} & \overline{\frac{\partial N^3}{\partial x_1}} & \overline{\frac{\partial N^3}{\partial x_2}} \\ \overline{\frac{\partial N^1}{\partial x_1}} & \overline{\frac{\partial N^1}{\partial x_2}} & \overline{\frac{\partial N^2}{\partial x_1}} & \overline{\frac{\partial N^2}{\partial x_2}} & \overline{\frac{\partial N^3}{\partial x_1}} & \overline{\frac{\partial N^3}{\partial x_2}} \cdots\cdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3D is also similar

Finally assemble stiffness & force vectors as before

$$[k^{el}] = \int_{\Omega_{el}} [\bar{B}]^T [D] [\bar{B}]\, dV \qquad \underset{\sim}{R} = \int_{\Omega_{el}} [\bar{B}]^T \underline{\sigma}\, dV$$

Can evaluate these with full integration

Notes:
- This is default element in ABAQUS
- This can be extended to finite strains

"F-bar" method: define deformation gradient

$$\bar{F}_{ij} = \frac{\bar{J}}{J} F_{ij} \qquad J = \det(F) \qquad \bar{J} = \frac{1}{\Omega_{el}} \int_{\Omega_{el}} J\, dV$$

6.5: "Hybrid" elements for fully incompressible solids

All methods dicussed so far give singular $[K]$ when $\nu \to 0.5$

Need to re-derive FE equations for fully incompressible linear elasticity

Field Eqs: $\varepsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$

$$\sigma_{ij} = 2\mu\, \varepsilon_{ij} + \frac{1}{3} p\, \delta_{ij}$$

$$\frac{1}{3}\varepsilon_{kk} = \frac{1}{3}\frac{\partial u_k}{\partial x_k} = 0 \quad \text{(Incompressibility)}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \quad \text{(Equilibrium)}$$

Weak form: Let η_i and q be "test functions"

Hence

$$\int_R \frac{\partial}{\partial x_j}\left\{2\mu\varepsilon_{ij} + \frac{1}{3}p\delta_{ij}\right\}\eta_i \, dV = 0 \qquad (1)$$

$$\int_R \frac{1}{3}\frac{\partial u_k}{\partial x_k} q \, dV = 0 \qquad (2)$$

Integrate (1) by parts:

$$\int_R \left\{2\mu\varepsilon_{ij}\frac{\partial \eta_i}{\partial x_j} + \frac{1}{3}p\frac{\partial \eta_i}{\partial x_j}\right\} dV - \int_{S_2} t_i^* \eta_i \, dA = 0 \quad \forall \eta_i \qquad (3)$$

(3), (2) are new governing eqs

FE interpolation :

$$u_i = N^a(\underline{x})\, u_i^a \qquad \eta_i = N^a \eta_i^a$$

$$p = M^a(\underline{x})\, p^a \qquad q = M^a(\underline{x})\, q^a$$

M, N - interpolation functions

u_i^a, η_i^a - nodal values

p^a, q^a - nodal values of pressure