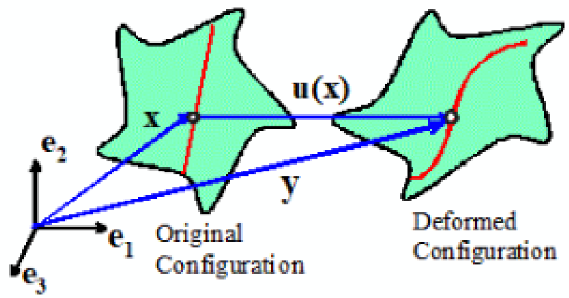
Review – FEA for finite deformation - hyperelasticity

Kinematics $y_i = x_i + u_i(x_k)$ $\quad F_{ij} = \delta_{ij} + \dfrac{\partial u_i}{\partial x_j}$

$J = \det(\mathbf{F}), \quad B_{ij} = F_{ik} F_{jk}$



Equilibrium

$$\int_R \sigma_{ji} \frac{\partial \eta_i}{\partial y_j} dV_0 - \int_{S_2} t_i \eta_i dA = 0 \quad \forall\, admiss\ \eta_i \Leftrightarrow \partial\sigma_{ij} / \partial y_i = 0 \;\; n_i \sigma_{ij} = t_j^* \;\; on \;\; S_2$$

Shear Modulus Bulk Modulus

Stress-strain relation

$$\sigma_{ij} = \frac{\mu_1}{J^{5/3}} \left(B_{ij} - \frac{1}{3} B_{kk} \delta_{ij} \right) + K_1 (J-1) \delta_{ij} \qquad \tau_{ij} = J \sigma_{ij}$$

Discrete equilibrium eq $\quad R_i^a[u_k^b] = f_i^a \quad R_i^a = \int_{V_0} \dfrac{\partial N^a}{\partial y_j} \tau_{ji} dV_0 \quad f_i^a = \int_{S_2^0} t_i^{0*} N^a dA_0$

Solve nonlinear equilibrium eq with Newton-Raphson

Newton correction equation $\quad K_{aibk} dw_k^b = f_i^a - R_i^a[w_k^b]$

$$K_{aibk} = \int_{V_0} \frac{\partial N^a}{\partial y_j} \frac{\partial \tau_{jl}}{\partial B_{pq}} \left(\delta_{pk} B_{ql} + \delta_{qk} B_{pl} \right) \frac{\partial N^b}{\partial y_l} dV_0 - \int_{V_0} \frac{\partial N^a}{\partial y_k} \tau_{ji} \frac{\partial N^b}{\partial y_j} dV_0$$

$$\frac{\partial \tau_{ij}}{\partial B_{kl}} = \frac{\mu_1}{J^{2/3}} \left(\frac{\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}}{2} - \frac{1}{3} \delta_{kl} \delta_{ij} \right) - \frac{1}{3} \frac{\mu_1}{J^{2/3}} \left(B_{ij} - \frac{1}{3} B_{mm} \delta_{ij} \right) B_{kl}^{-1} + K_1 (J^2 - \frac{1}{2} J) \delta_{ij} B_{kl}^{-1}$$

FE implementation

$$[K^{el}] = \int_{\Omega_{e0}} [B]^T [D] [G] [B^*] \, dV_0 - \int_{\Omega_{e0}} [Y] \, dV_0$$

$[B]$ - usual $[B]$ (6×3 NNODES) - except $\partial N / \partial \underline{x}$ now $\partial N / \partial y$

$$[D] = \begin{bmatrix} \partial \tau_{11} / \partial B_{11} & \partial \tau_{11} / \partial B_{22} & \partial \tau_{33} / \partial B_{33} \\ \partial \tau_{22} / \partial B_{11} & & \\ \vdots & & \end{bmatrix}$$

Not symmetric

Define $\underline{b} = [\, B_{11} \;\; B_{22} \;\; B_{33} \;\; B_{12} \;\; \cdots \,]$

$$\underline{b}^{-1} = [\, B_{11}^{-1} \;\; B_{22}^{-1} \;\; \cdots \,]$$

$$\underline{I} = [1, 1, 1, 0, 0, 0]$$

Then:

$$\mathbf{D} = \frac{\mu_1}{J^{2/3}} \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1/2 & & \\ & & & & 1/2 & \\ 0 & & & & & 1/2 \end{bmatrix} + \frac{\mu_1}{3J^{2/3}} \left(\frac{B_{nn}}{3} \underline{I} \otimes \underline{b}^{-1} - \underline{I} \otimes \underline{I} - \underline{b} \otimes \underline{b}^{-1} \right) + K_1 J (J - 1/2) \underline{I} \otimes \underline{b}^{-1}$$

[B*] is a modified [B] that maps $\underline{u}^{el}$ onto displacement grads

$$\underline{g} = [B^*] \, \underline{u}^{el}$$

$$\underline{g} = \left[\frac{\partial u_1}{\partial y_1} \quad \frac{\partial u_2}{\partial y_2} \quad \frac{\partial u_3}{\partial y_3} \quad \frac{\partial u_1}{\partial y_2} \quad \frac{\partial u_2}{\partial y_1} \quad \frac{\partial u_1}{\partial y_3} \quad \frac{\partial u_3}{\partial y_1} \quad \frac{\partial u_2}{\partial y_3} \quad \frac{\partial u_3}{\partial y_2} \right]^T$$

Then $[B^*]$ has form

$$
\mathbf{B}^* = \begin{bmatrix}
\frac{\partial N^1}{\partial y_1} & 0 & 0 & \frac{\partial N^2}{\partial y_1} & 0 & 0 \\
0 & \frac{\partial N^1}{\partial y_2} & 0 & 0 & \frac{\partial N^2}{\partial y_2} & 0 \\
0 & 0 & \frac{\partial N^1}{\partial y_3} & 0 & 0 & \frac{\partial N^2}{\partial y_3} \\
\frac{\partial N^1}{\partial y_2} & 0 & 0 & \frac{\partial N^2}{\partial y_2} & 0 & 0 \\
0 & \frac{\partial N^1}{\partial y_1} & 0 & 0 & \frac{\partial N^2}{\partial y_1} & 0 & \cdots\cdots \\
\frac{\partial N^1}{\partial y_3} & 0 & 0 & \frac{\partial N^2}{\partial y_3} & & \\
0 & 0 & \frac{\partial N^1}{\partial y_1} & 0 & 0 & \frac{\partial N^2}{\partial y_1} \\
0 & \frac{\partial N^1}{\partial y_3} & 0 & & & \\
0 & 0 & \frac{\partial N^1}{\partial y_2} & & &
\end{bmatrix}
$$

$$
\mathbf{G} = \begin{bmatrix}
2B_{11} & 0 & 0 & 2B_{12} & 0 & 2B_{13} & 0 & 0 & 0 \\
0 & 2B_{22} & 0 & 0 & 2B_{12} & 0 & 0 & 2B_{23} & 0 \\
0 & 0 & 2B_{33} & 0 & 0 & 0 & 2B_{13} & 0 & 2B_{13} \\
2B_{12} & 2B_{12} & 0 & 2B_{22} & 2B_{11} & 2B_{23} & 0 & 2B_{13} & 0 \\
2B_{13} & 0 & 2B_{13} & 2B_{23} & 0 & 2B_{33} & 2B_{11} & 0 & 2B_{12} \\
0 & 2B_{23} & 2B_{23} & 0 & 2B_{13} & 0 & 2B_{12} & 2B_{33} & 2B_{22}
\end{bmatrix}
$$

Geometric Stiffness

$$
\mathcal{Y}_{aibk} = \frac{\partial N^a}{\partial y_k} \tau_{ji} \frac{\partial N^b}{\partial y_j}
$$

Define $P_{ak} = \dfrac{\partial N^a}{\partial y_k}$ $\qquad S_{bi} = \underbrace{\tau_{ji} \dfrac{\partial N^b}{\partial y_j}}$

$n \times 3 \qquad\qquad n\times 3$

Then

$$[Y] = \begin{bmatrix} P_{11} S_{11} & P_{12} S_{11} & P_{13} S_{11} & P_{11} S_{21} \cdots \\ P_{11} S_{12} \\ P_{11} S_{13} \\ P_{21} S_{11} \end{bmatrix}$$

This is

elemental product

$$[Y] = \begin{bmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{11} & P_{12} & P_{13} & \cdots \\ P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \end{bmatrix} \circledast \begin{bmatrix} S_{11} & S_{11} & S_{11} & S_{21} & \cdots \\ S_{12} & S_{12} & S_{12} \\ S_{13} & S_{13} & S_{13} \end{bmatrix}$$

We can assemble [P], [S] with "spread", reshapes

$$\begin{bmatrix} P_{11} & P_{11} & \text{copy} \\ P_{12} & P_{12} \cdots & \text{nnode} \\ P_{13} & P_{13} & \text{times} \\ & & \text{Spread} \end{bmatrix} \xrightarrow{\text{Reshape}} \underbrace{[P_{11}\ P_{12}\ P_{13}\ P_{11} \cdots]}_{\text{Pvec}} \rightarrow \begin{bmatrix} P_{11} & P_{12} & P_{13} \cdots \\ \downarrow & & \\ \text{copy twice} & & \\ \text{(spread)} & & \end{bmatrix}$$

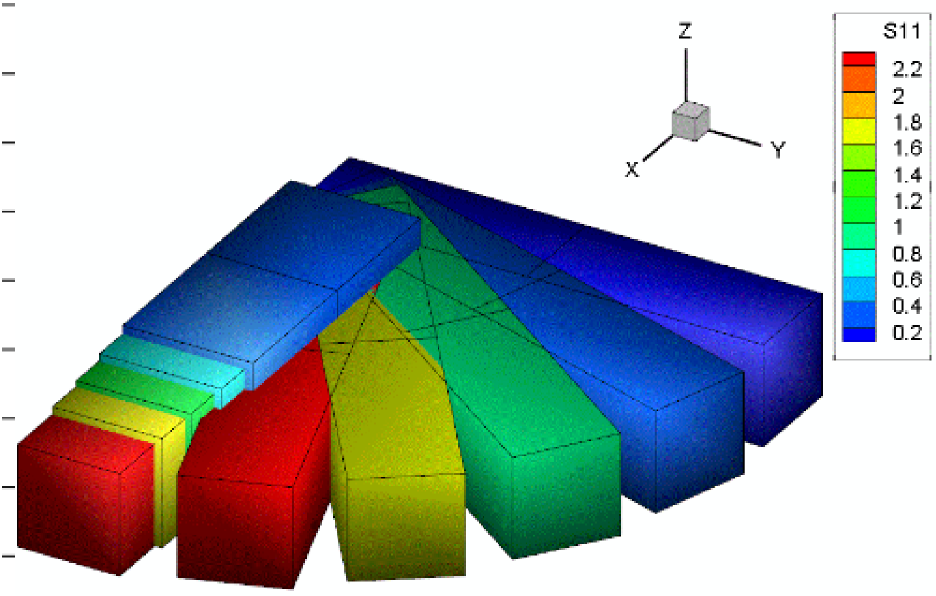
```fortran
!       Geometric stiffness

        S = reshape(matmul(transpose(Bbar),stress),(/3,3*NNODE/3/))
        do i = 1,NNODE
          Pvec(1:3*NNODE) = reshape(spread(transpose(dNdy(i:i,1:3)),
                                   dim=2,ncopies=NNODE),(/3*NNODE/))
          Pmat(3*i-2:3*i,1:3*NNODE) = spread(Pvec,dim=1,ncopies=3)
          Svec(1:3*NNODE) = reshape(spread(S(1:3,i:i),
                                   dim=2,ncopies=NNODE),(/3*NNODE/))
          Smat(3*i-2:3*i,1:3*NNODE) = spread(Svec,dim=1,ncopies=3)
        end do

        AMATRX(1:3*NNODE,1:3*NNODE) = AMATRX(1:3*NNODE,1:3*NNODE) -
          Pmat(1:3*NNODE,1:3*NNODE)*transpose(Smat(1:3*NNODE,1:3*NNODE))
                            *w(kint)*determinant
```

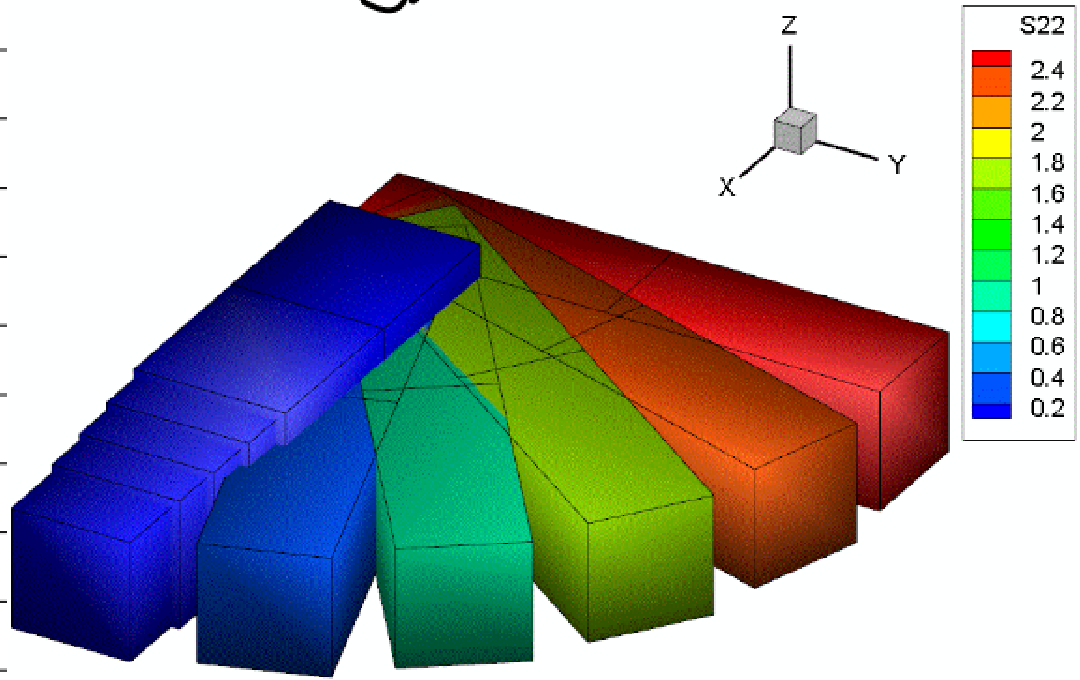
← Pvec

← Copy to create 3 rows of [P]

← Elemental product

Sample Results - Stretch / Rotate a hyperelastic bar





Remarks: For actual applications elastomers are near incompressible ⇒ B-bar or hybrid elements

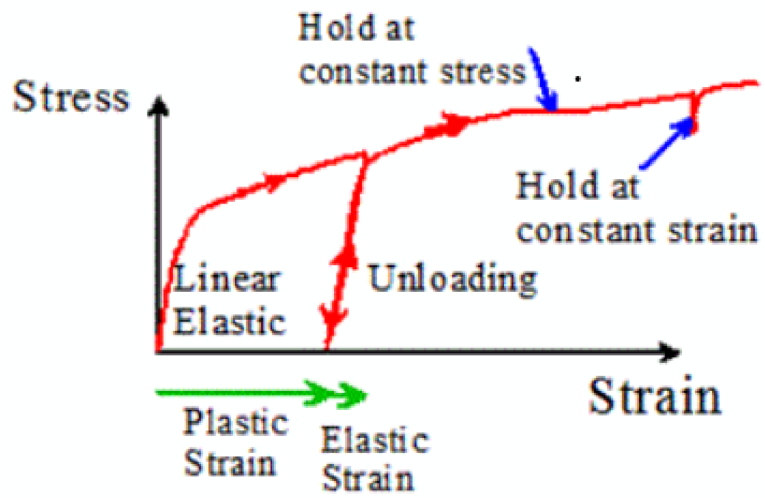
See HW #6 2015 for B-bar implementation

In ABAQUS best not to use UMAT for finite strain elasticity – use UHYPER instead

7.3) FEA for nonlinear & history dependent materials

- small strain viscoplasticity

Assume infinitesimal deformations



Constitutive law

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{e} + \dot{\varepsilon}_{ij}^{p}$$

Elastic strains

$$\dot{\varepsilon}_{ij}^{e} = \frac{1+\nu}{E}\dot{S}_{ij} + \frac{1-2\nu}{E}\dot{\sigma}_{kk}\frac{\delta_{ij}}{3}$$

$$S_{ij} = \sigma_{ij} - \sigma_{kk}\,\delta_{ij}/3$$

Plastic strain rate:

(1) Isotropic material
(2) Plastic strains volume preserving
(3) Assume plastic strain rate parallel to deviatoric stress "associated flow"

Let $\sigma_e = \sqrt{\frac{3}{2} S_{ij} S_{ij}}$ Von Mises stress

$$\dot{\varepsilon}_{ij}^{p} = \dot{\varepsilon}_0 \left(\frac{\sigma_e}{\sigma_0} \right)^m \frac{3}{2} \frac{S_{ij}}{\sigma_e}$$ $\dot{\varepsilon}_0$, m material props

$$\sigma_0 = Y \left(1 + \frac{\varepsilon_e}{\varepsilon_0} \right)^{1/n}$$ Y, n, ε_0 "

ε_e = accumulated plastic strain magnitude

$\varepsilon_e = 0$ @ $t = 0$ $\dot{\varepsilon}_e = \sqrt{2 \dot{\varepsilon}_{ij}^{p} \dot{\varepsilon}_{ij}^{p} / 3}$

FEA implementation :

(1) Time dependent material

(2) History dependent - need to track load history

Address these with simple time-stepping

(1) Assume $u_i = 0$ $\varepsilon_e = 0$ @ $t = 0$

(2) Consider time increment Δt

(3) Find Δu_i , $\Delta \varepsilon$ by satisfying equilibrium at end of step

$u_i \rightarrow u_i + \Delta u_i$ $\varepsilon_e \rightarrow \varepsilon_e + \Delta \varepsilon_e$

Repeat.

Equilibrium equation (weak form) has usual form

$$\int_R \sigma_{ij}\left[\Delta\varepsilon_{pq}(\Delta u_k), \Delta\varepsilon_e, \Delta t\right]\frac{\partial \eta_i}{\partial x_j}\,dV - \int_{S_2} t_i^* \eta_i = 0 \quad \forall \text{ admiss } \eta_i$$

$$u_i + \Delta u_i = u_i^*(t+\Delta t) \quad \text{on } S_1$$