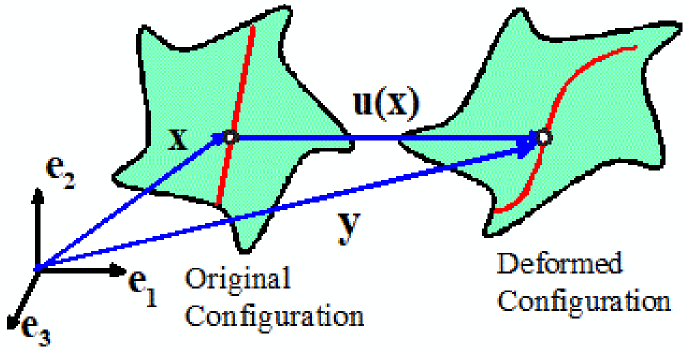
Review – Newmark time integration



Equation of motion $$\mathbf{M}\ddot{\mathbf{u}} = -\mathbf{R}(\mathbf{u}) + \mathbf{F}(t)$$

Integrate WRT time using time stepping:

(1) Initialize $$\mathbf{u}^0 = \mathbf{u}(\mathrm{t}=0) \qquad \mathbf{v}^0 = \frac{d\mathbf{u}}{dt}(\mathrm{t}=0)$$

$$\mathbf{a}^0 = -\mathbf{R}(\mathbf{u}^0) + \mathbf{F}(0)$$

Newmark Loop

(1) Solve $$\mathbf{M}\mathbf{a}^{n+1} = -\mathbf{R}\left(\mathbf{u}^n + \Delta t\mathbf{v}^n + \frac{\Delta t^2}{2}\left[(1-\beta_2)\mathbf{a}^n + \beta_2\mathbf{a}^{n+1}\right]\right) + \mathbf{F}(t_{n+1})$$

(2) Update $$\mathbf{v}^{n+1} = \mathbf{v}^n + \Delta t\left[(1-\beta_1)\mathbf{a}^n + \beta_1\mathbf{a}^{n+1}\right]$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\mathbf{v}^n + \frac{\Delta t^2}{2}\left[(1-\beta_2)\mathbf{a}^n + \beta_2\mathbf{a}^{n+1}\right]$$

Stability of Newmark algorithm

Examine Newmark solution to harmonic oscillator

$$\ddot{x} + \omega^2 x = 0 \qquad (1)$$

(ω represents highest frequency mode in mesh)

Recall $x(t) = C e^{i\omega t}$ C depends on initial condition

Find solution to (1) with Newmark

Note $a(t) = -\omega^2 x(t)$

$$x_{n+1} = x_n + V_n \Delta t + \tfrac{1}{2}\Delta t^2 \left\{ -(1-\beta_2)\omega^2 x_n - \beta_2 \omega^2 x_{n+1} \right\}$$

$$V_{n+1} = V_n + \Delta t \left\{ -(1-\beta_1)\omega^2 x_n - \beta_1 \omega^2 x_{n+1} \right\}$$

Solve for x_{n+1} $\quad v_{n+1}/\omega \quad\quad \Omega = \omega \Delta t$

$$\underbrace{\begin{bmatrix} 1+\Omega^2\beta_2/2 & 0 \\ \beta_1 \Omega & 1 \end{bmatrix}}_{[P]} \underbrace{\begin{bmatrix} x_{n+1} \\ \frac{v_{n+1}}{\omega} \end{bmatrix}}_{q_{n+1}} = \underbrace{\begin{bmatrix} 1-(1-\beta_2)\Omega^2/2 & \Omega \\ -(1-\beta_1)\Omega^2 & 1 \end{bmatrix}}_{[S]} \underbrace{\begin{bmatrix} x_n \\ \frac{v_n}{\omega} \end{bmatrix}}_{q_n}$$

$$q_{n+1} = [A]\, q_n \qquad [A] = [P]^{-1}[S]$$

General Solution

$$q_n = C_1 \underline{Q}_1 \lambda_1^n + C_2 \underline{Q}_2 \lambda_2^n$$

$\{\lambda_1, \lambda_2, \underline{Q}_1, \underline{Q}_2\}$ are (complex) eigenvalues / eigenvectors of $[A]$

C_1, C_2 are constants

Now examine character of solution.

Note: $x(t)$ is harmonic $\Rightarrow$ λ_i must be complex for newmark to be accurate

For energy conservation $|\lambda_i|$ must equal 1

$$\lambda = \frac{4+\Omega^2(2\beta_2-2\beta_1-1) \pm \Omega\sqrt{\Omega^2\left(4\beta_1^2+4\beta_1-8\beta_2+1\right)-16}}{2(2+\beta_2\Omega^2)}$$

Implications: Explicit Dynamics $\beta_2 = 0$

For complex λ: $\Omega^2 \leq \dfrac{16}{1+4\beta_1(1+\beta_1)}$

Note $|\lambda| = 1 + \Omega^2\left(\tfrac{1}{2} - \beta_1\right)$

Hence for energy conservation $\beta_1 = \frac{1}{2}$

For harmonic solution $\Omega^2 = (\omega \Delta t)^2 \le 4$

Conditional stability $\Rightarrow$ $\Delta t \le \frac{2}{\omega}$

Best choice for β_1 is $\beta_1 = \frac{1}{2}$

<u>Implicit</u> : $\beta_1 = \frac{1}{2}$ to conserve energy

With this choice $\Omega^2 \le \dfrac{4}{1-2\beta_2}$ for complex roots

Choose $\beta_2 = \frac{1}{2}$ $\Rightarrow$ unconditionally complex

Choose $\beta_1 = \beta_2 = \frac{1}{2}$ for best accuracy

More generally if λ_i are real

— Solution stable if $\max_i \{\lambda_i\} \leq 1$

Unstable otherwise

We need $\beta_1 \geq \frac{1}{2}$, $\beta_2 \geq \beta_1$ to satisfy this condition

Mass scaling

A typical engineering problem there are many time-scales

Shortest time scale: time for elastic wave to propagate through solid

Time dependent materials (eg viscoelasticity, plasticity, etc) introduce additional time scales

We often are interested in resolving only material time scales; elastic waves are of no interest

We can speed up Newmark using mass scaling

ie $\gamma [M] \ddot{\underline{u}} = \underline{F}(t) - \underline{R}[\underline{u}]$ $\quad \gamma > 1$ is mass scaling factor

Choose γ so that KE remains negligible compared to plastic or viscoelastic dissipation

8.3 Modal Time Integration

Method of integrating (wrt time) EOM for linear elastic solid

EOM are $[M]\,\ddot{\underline{u}} = \underline{F}(t) - \underline{R}(\underline{u})$

For linear elasticity $\underline{R} = [K]\,\underline{u}$ $\quad [K]$ is const

$$[M]\,\ddot{\underline{u}} + [K]\,\underline{u} = \underline{F}(t)$$

(forced undamped spring-mass system)

Re-write EOM as

$$\underbrace{[M]^{1/2}\,\ddot{\underline{u}}}_{\underline{W}} + [M]^{-1/2}[K]\,\underbrace{\underline{u}}_{[M]^{-1/2}\,\underline{W}} = [M]^{-1/2}\,\underline{F}(t)$$

$$\ddot{\underline{W}} + [H]\,\underline{W} = [M]^{-1/2}\,\underline{F}(t)$$

$[H] = [M]^{-1/2}\,[K]\,[M]^{-1/2}$ $\quad$ $[H]$ is positive semi-definite & symmetric

Eigenvalues of $[H] = \omega_i^2$ $\quad$ (for N DOF we get N eigenvalues)

Eigenvectors (normalized) q_i satisfy $q_i \cdot q_j = \delta_{ij}$

We can decompose $[H]$ as $\sum_i \omega_i^2\, q_i \otimes q_i$

We can also project $\underline{W}$ onto q_i

$$\underline{W} = \sum_i (\underline{W} \cdot q_i)\, q_i$$

Re-write EOM for $\underline{W}$ as

$$\ddot{\underline{W}} + \left[\sum_i \omega_i^2 \, q_i \otimes q_i\right] \underline{W} = [M]^{-1/2} \underline{F}(t)$$

Dot with q_j: on both sides:

$$\underbrace{q_j \cdot \ddot{\underline{W}}} + \underbrace{\omega_j^2 \, [q_j \cdot \underline{W}]} = q_j \, [M]^{-1/2} F(t)$$

$$\ddot{r}_j(t) + \omega_j^2 \, r_j(t) = f_j(t)$$

EOM for $r_j(t)$ is eom for a forced harmonic oscillator

Initial Conditions: $r_j(0) = q_j \cdot [m]^{1/2} \underline{u}(0)$

$$\dot{r}_j(0) = q_j \cdot [m]^{1/2} \underline{V}(0)$$

Example: Suppose $f_j(t) = A_j \sin \omega t$

$$r_j(t) = \frac{A_j \sin \omega t}{\omega^2 - \omega_j^2} + r_j(0) \cos \omega_j t + \frac{\dot{r}_j(0)}{\omega_j} \sin \omega_j t$$

Finally given $r_j(t)$ we can reconstruct $\underset{\sim}{u}(t)$

$$\underset{\sim}{u}(t) = [M]^{-1/2} \sum_j r_j(t) \, \underset{\sim}{q}_j$$

This is a "modal" solution for $\underset{\sim}{u}$

$\underset{\sim}{u}_j^* = [M]^{-1/2} \underset{\sim}{q}_j$ are mode shapes

ω_j are natural freqs

We usually truncate the sum for some reasonable number of modes

$$\underline{u}(t) \approx \sum_{j=1}^{\text{truncated set}} r_j(t)\, \underline{u}_j^*$$

Available in ABAQUS under "Linear Perturbation" step