11 Interfaces and Contact

11.1 "Cohesive Zone" models of fracture

Goal: model crack nucleation and growth along a weak interface

Approach: Introduce plane $\Gamma$ where solid can separate

Introduce additional constitutive equations that specify tractions acting on material planes as functions of separation
Cohesive Zone Laws

Kinematics & Kinetics

Let \( \Delta n = (u^+ - u^-) \cdot n \)

\[ \Delta \alpha = (u^+ - u^-) \cdot e^\alpha \]

Also \( T_n = e^- \cdot n \)

\[ T_\alpha = e^- \cdot e^\alpha \]

We usually assume \( \Delta n, \Delta \alpha \) are small

For large relative displacement use contact elements

Constitutive laws must give \( \{T_n, T_\alpha\} = f(\Delta n, \Delta \alpha) \)
Examples: Reversible cohesive zone law

Let $\Phi(\Delta_n, \Delta_t)$ be work done per unit area to separate planes.

$$T_n = \frac{\partial \Phi}{\partial \Delta_n}, \quad T_\alpha = \frac{\partial \Phi}{\partial \Delta_\alpha}$$

**Example:** Xu–Needleman

$$\Phi(\Delta_n, \Delta_t) = \phi_n - \phi_n \left(1 + \frac{\Delta_n}{\delta_n}\right) \exp\left(-\frac{\Delta_n}{\delta_n}\right) \exp\left(-\frac{\beta^2 \Delta_t^2}{\delta_n^2}\right)$$

$\phi_n =$ interface energy

$\delta_n, \beta =$ material props controlling compliance
Hence

\[ T_n = \sigma_{\text{max}} \frac{\Delta_n}{\delta_n} \exp\left(1 - \frac{\Delta_n}{\delta_n}\right) \]

\[ T_\alpha = 2\sigma_{\text{max}} \left(\frac{\beta^2 \Delta_\alpha}{\delta_n}\right) \left(1 + \frac{\Delta_n}{\delta_n}\right) \exp\left(1 - \frac{\Delta_n}{\delta_n}\right) \exp\left(-\frac{\beta^2 \Delta_\alpha^2}{\delta_n^2}\right) \]

\[ \sigma_{\text{max}} = \frac{\phi_n}{\delta_n} \exp(1) \]

\[ T_{\text{max}} = \beta^2 \sigma_{\text{max}} \]

Notes:

1. Interface is reversible: Heals if interfaces contact after separation

2. Interface can behave strangely under large pressure
Irreversible Cohesive Zone Laws

Assumptions:

1. No inter-penetration \( \Delta_n > 0 \)

2. Irreversible behavior in tension
   - Undamaged interface has stiffness \( k_0 = \sigma_{\text{max}} \frac{d_1}{d} \)
   - If separation exceeds \( d_1 \), interface stiffness drops

Damage is quantified by an internal variable \( 0 < D < 1 \)

- Undamaged solid has \( D = 0 \)
- Total failure \( D = 1 \)
To quantify magnitude of separation and traction

\[ \lambda = \sqrt{\Delta_{n}^{2} + \beta^{2} (\Delta_{1}^{2} + \Delta_{2}^{2})} \]

\[ \tau = \sqrt{T_{n}^{2} + (T_{1}^{2} + T_{2}^{2}) / \beta^{2}} \]

Assume that interface potential \( \Phi(\lambda, \Delta) \) has form

\[ \Phi = \frac{1}{2} k_{0} (1-\delta) \lambda^{2} \]

Hence \( T_{n} = \frac{\partial \Phi}{\partial \Delta_{n}} = k_{0} (1-\delta) \Delta_{n} \)

\( T_{\alpha} = \frac{\partial \Phi}{\partial \Delta_{\alpha}} = \beta^{2} k_{0} (1-\delta) \Delta_{\alpha} \)
We also need an evolution equation for $D$. We have that

$$R_0(1-D) \lambda = \Omega_{\text{max}} \left( 1 - \frac{\lambda - d_1}{d_2 - d_1} \right) = \Omega_{\text{max}} / d_1$$

Take time derivative

$$\frac{dD}{dt} = \begin{cases} 0 & \lambda < \frac{d_1 d_2}{(1-D) d_2 + D d_1} \quad \text{or} \quad \frac{d\lambda}{dt} < 0 \quad \text{or} \quad D = 1 \\
\left( 1-D + \frac{d_1}{d_2 - d_1} \right) \frac{1}{\lambda} \frac{d\lambda}{dt} & \text{Otherwise} 
\end{cases}$$

History dependent - $D$ is a state variable.
Viscous Regularization

Problem: Crack nucleation can be an unstable process

Example: Xn- Needleman interface between elastic blocks in uniaxial tension

\[
\frac{\sigma}{\sigma_{\text{max}}} = \frac{\Delta n}{\delta n} \exp\left(1 - \frac{\Delta n}{\delta n}\right)
\] 

\[
U_0 = \frac{\sigma}{E} a + \Delta n \Rightarrow \frac{U_0}{E} = \frac{\sigma}{\sigma_{\text{max}}} a + \Delta n \frac{E \delta n}{\sigma_{\text{max}} a}
\]
Stress-displacement relation

For large $\Lambda$, we see stable separation as $U_0$ increases.

For small $\Lambda$, we see a snap from A-B.

For a displacement controlled static analysis, there is no equilibrium path from A-B. Newton iterations will fail at A.

We can fix this by adding some viscosity to the interface.
For example we could use

\[ T_n = \frac{d\Phi}{d\Delta n} + \eta \frac{d\Delta n}{dt} \quad \{ \text{\(\eta\) is a small viscosity} \} \]

\[ T_\alpha = \frac{d\Phi}{d\Delta \alpha} + \eta \frac{d\Delta \alpha}{dt} \quad \{ \text{Choice of \(\eta\) dictated by computational cost & accuracy} \} \]
FE implementation

Assume small strain linear elasticity

Governing Equations

\[ \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \]
\[ \sigma_{ij} = C_{ijkl} \frac{\partial u_k}{\partial x_l} \]

Boundary Conditions: \( \sigma_{ij} n_j = e_i^* \) on \( S_e \)
\[ u_i = u_i^* \) on \( S_i \)

Interface Relations: \( n_i \sigma_{ij} n_j = T_n \) on \( \Gamma^- \)
\[ = -T_n \) on \( \Gamma^- \)
\[ e_i \sigma_{ij} n_j = \tau \) on \( \Gamma^- \)
\[ = -\tau \) on \( \Gamma^- \)
Constitutive eqs for cohesive zone $(T_n, T_a) = f(\Delta n, \Delta a)$

We need a new weak form for equilibrium

Let $\eta_i$ be a kinematically admissible displacement field

Note $\eta_i$ may be discontinuous across $\Gamma$

Let $\eta_i^+$, $\eta_i^-$ denote $\eta_i$ on two sides of $\Gamma$

Weak form of equilibrium

\[
\int \frac{\partial}{\partial x_j} \eta_i \text{ } dV = 0 \Rightarrow \int_{\Omega} \left\{ \frac{\partial}{\partial x_j} (\sigma_{ij} \eta_i) - \delta_{ij} \frac{\partial \eta_i}{\partial x_j} \right\} dV = 0
\]

Apply divergence theorem
\[ \int_{\Gamma} \sigma_{ij} n_j \eta_i \, dA - \int_{\Gamma^+} \sigma_{ij} n_j \eta^+_i \, dA + \int_{\Gamma^-} \sigma_{ij} n_j \eta^-_i \, dA \\
- \int_{\mathcal{R}} \sigma_{ij} \frac{d\eta_j}{dx_i} \, dV = 0 \forall \text{ admiss } \eta_i \]

Note on \( \Gamma \): \( \sigma_{ij} n_j = T_n n_i + T_{\alpha} e_i^\alpha \)

\[ \int_{\mathcal{R}} \sigma_{ij} \frac{d\eta_j}{dx_i} \, dV + \int_{\Gamma} (\eta^+_i - \eta^-_i) n_i \, dA \\
+ \int_{\Gamma^-} T_{\alpha} (\eta^+_i - \eta^-_i) e_i^\alpha \, dA \\
- \int_{\mathcal{S}_e} e_i^+ \eta_i \, dA = 0 \forall \text{ admiss } \eta_i \]
This looks like usual PWR but with some additional terms on interface

Introduce usual interpolations inside volume

\[ U_i = N^a U_i^a \quad \eta_i = N^a \eta_i^a \]

Account for displacement discontinuity using cohesive zone elements

Generate meshes inside two solids joined by interface with coincident nodes on \( \Gamma \)