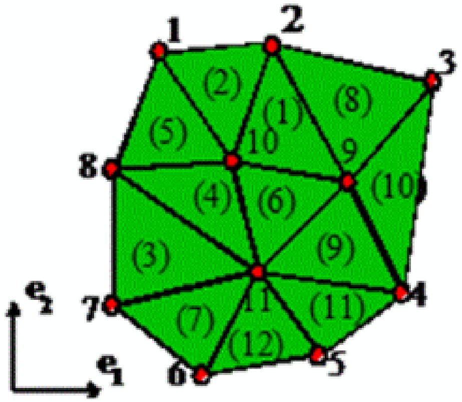
Review – Simple FEA for plane linear elasticity

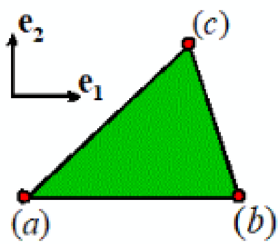
Background: Governing equations for linear elasticity
The Principle of Minimum Potential Energy



- Plane strain linear elasticity problem as an energy minimization

$$\Phi = \int_A \phi dA - \int_{S_2} \mathbf{t}^* \cdot \mathbf{u} ds$$

- Interpolating the displacement field using constant strain triangles
- Computing the potential energy
  - Strains, stresses and strain energy density inside an element. Element stiffness matrix
  - Potential energy of a loaded element face – element force vector
  - Total potential energy – global stiffness and global force
  - Constrained boundary conditions



- Minimizing the total potential energy

$$\Phi = \frac{1}{2}\left(\mathbf{u}^{Global}\right)^T [\mathbf{K}]\mathbf{u}^{Global} - \left(\mathbf{u}^{Global}\right)^T \cdot \mathbf{f}^{Global} \quad \Rightarrow \quad [K]\mathbf{u}^{Global} = \mathbf{f}^{Global}$$

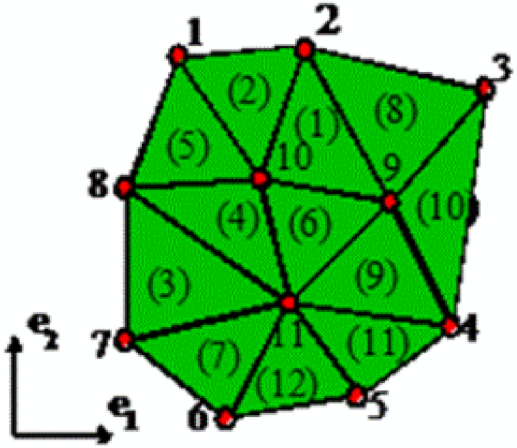
- Implementation
  - Data structures for mesh and BC definition
  - Assembling the element and global stiffness matrices
  - Prescribing boundary conditions
  - Solution and post-processing

Review

Interpolating displacements

(Constant strain triangles – many other interpolation schemes exist)

$$\mathbf{u} = N^a(\mathbf{x})\mathbf{u}^a + N^b(\mathbf{x})\mathbf{u}^b + N^c(\mathbf{x})\mathbf{u}^c$$

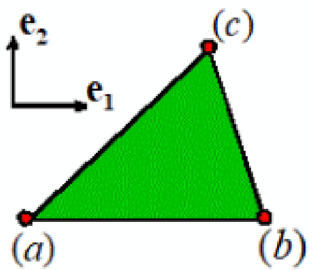


Interpolation Functions

$$N^a(x_1,x_2) = \frac{\left(x_2 - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1 - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}{\left(x_2^{(a)} - x_2^{(b)}\right)\left(x_1^{(c)} - x_1^{(b)}\right) - \left(x_1^{(a)} - x_1^{(b)}\right)\left(x_2^{(c)} - x_2^{(b)}\right)}$$

$$N^b(x_1,x_2) = \frac{\left(x_2 - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1 - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}{\left(x_2^{(b)} - x_2^{(c)}\right)\left(x_1^{(a)} - x_1^{(c)}\right) - \left(x_1^{(b)} - x_1^{(c)}\right)\left(x_2^{(a)} - x_2^{(c)}\right)}$$

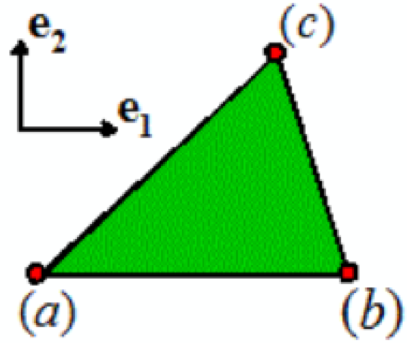
$$N^c(x_1,x_2) = \frac{\left(x_2 - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1 - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}{\left(x_2^{(c)} - x_2^{(a)}\right)\left(x_1^{(b)} - x_1^{(a)}\right) - \left(x_1^{(c)} - x_1^{(a)}\right)\left(x_2^{(b)} - x_2^{(a)}\right)}$$



Topics for today's class

- Calculating strains
- Calculating the strain energy density
- Calculating the potential energy of external forces
- Minimizing the PE
- Imposing constrained displacements
- Solution and post-processing

Calculating strains (Plane strain)



$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\underline{u} = N^a \underline{u}^a + N^b \underline{u}^b + N^c \underline{u}^c$$

$$\Rightarrow \varepsilon_{11} = \frac{\partial N^a}{\partial x_1} u_1^a + \frac{\partial N^b}{\partial x_1} u_1^b + \frac{\partial N^c}{\partial x_1} u_1^c$$

Re-write as matrix operation

Let $\quad \underline{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}]$

$$\underline{u}^{el} = [u_1^a \; u_2^a, u_1^b, u_2^b, u_1^c, u_2^c]$$

Now $\underset{\sim}{\varepsilon} = [B]\, \underset{\sim}{u}^{el}$ $\quad$ $[B]$ maps $\underline{u}^{el} \to \underline{\varepsilon}$

$$[B] = \begin{bmatrix} \frac{\partial N^a}{\partial x_1} & 0 & \frac{\partial N^b}{\partial x_1} & 0 & \frac{\partial N^c}{\partial x_1} & 0 \\ 0 & \frac{\partial N^a}{\partial x_2} & 0 & \frac{\partial N^b}{\partial x_2} & 0 & \frac{\partial N^c}{\partial x_2} \\ \frac{\partial N^a}{\partial x_2} & \frac{\partial N^a}{\partial x_1} & \frac{\partial N^b}{\partial x_2} & \frac{\partial N^b}{\partial x_1} & & \end{bmatrix}$$

Note that $[B]$ is constant for 3 noded triangle

Strain energy density $\phi = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$

$= (\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + 2\varepsilon_{12}\sigma_{12})/2$

Define $\underset{\sim}{\sigma} = [\sigma_{11} \quad \sigma_{22} \quad \sigma_{12}]$

$\Rightarrow \phi = \frac{1}{2}\underset{\sim}{\varepsilon}^T\underset{\sim}{\sigma}$

Recall $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Matrix form for plane strain $\underset{\sim}{\sigma} = [D]\underset{\sim}{\varepsilon}$

$$[D] = \frac{E}{(1-2\nu)(1+\nu)} \underbrace{\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}}_{\text{Symmetric}}$$

Hence $\quad \phi = \frac{1}{2} \underline{\varepsilon}^T [D] \underline{\varepsilon}$

$$= \frac{1}{2} \underline{u}^{elT} \underbrace{[B]^T [D] [B]}_{\text{Symmetric \& constant}} \underline{u}^{el}$$

<u>Total strain energy of 1 element</u>

$$W^{el} = \int_{A_{el}} \phi \, dA = A_{el} \phi$$

Hence introduce element stiffness matrix $[k^{el}]$

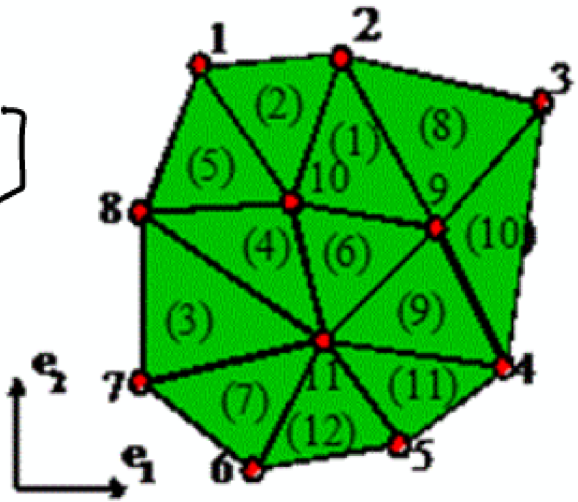
$$[k^{el}] = A_{el} [B^T] [D] [B] \leftarrow 6 \times 6$$

$$W^{el} = \frac{1}{2} \underline{u}^{elT} [k^{el}] \underline{u}^{el}$$

Total Strain Energy

$$W = \sum_{\text{elements}} W^{el} = \frac{1}{2}\sum_{el} \underset{\sim}{u}^{el\,T} [k^{el}] \underset{\sim}{u}^{el}$$

Global stiffness matrix



Define $\underset{\sim}{u} = [u_1^1\ u_2^1,\ u_1^2\ u_2^2\ \ldots\ u_1^N\ u_2^N]$

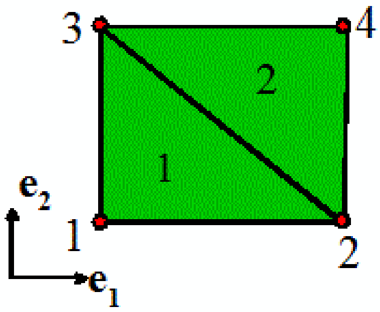
Now set

$$W = \frac{1}{2}\underset{\sim}{u}^T \underbrace{[K]}_{} \underset{\sim}{u}$$

"Global" stiffness , symmetric

Assembling [K] - consider 2 el mesh

Strain energy



$$W = \frac{1}{2}\begin{bmatrix} u_1^{(1)} & u_2^{(1)} & u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} \end{bmatrix} \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & \cdots & k_{16}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & & \\ \vdots & & \ddots & \\ k_{61}^{(1)} & & & k_{66}^{(1)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \end{bmatrix}$$

$$+ \; + \frac{1}{2}\begin{bmatrix} u_1^{(2)} & u_2^{(2)} & u_1^{(3)} & u_2^{(3)} & u_1^{(4)} & u_2^{(4)} \end{bmatrix} \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} & \cdots & k_{16}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & & \\ \vdots & & \ddots & \\ k_{61}^{(2)} & & & k_{66}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$

We can add missing terms from $\underline{u}$ to each $\underline{u}^{el}$

(add zeros into k^{el})

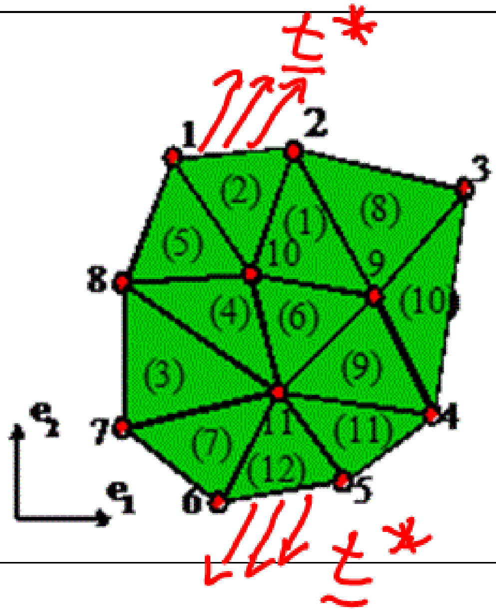
$$W = \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}^T \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & & \cdots & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{32}^{(1)} & & & & 0 & 0 \\ & & k_{33}^{(1)} & k_{34}^{(1)} & & & 0 & 0 \\ & & k_{43}^{(1)} & k_{44}^{(1)} & & & 0 & 0 \\ & & k_{53}^{(1)} & & & & 0 & 0 \\ & & \vdots & & & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & k_{11}^{(2)} & k_{12}^{(2)} & & & & \\ 0 & 0 & k_{21}^{(2)} & k_{22}^{(2)} & & & & \\ 0 & 0 & k_{31}^{(2)} & & & & & \\ 0 & 0 & & & & \ddots & & \\ 0 & 0 & \vdots & & & & & k_{56}^{(2)} \\ 0 & 0 & & & & & k_{65}^{(2)} & k_{66}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$

Factor $\underline{u}$

$$W = \frac{1}{2}\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}^T \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} & \cdots & & \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{32}^{(1)} & & & & \\ & & k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & & & \\ & & k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & & & \\ & & k_{53}^{(1)} + k_{31}^{(2)} & & & & \\ & & & & \ddots & & \\ & & \vdots & & & & k_{56}^{(2)} \\ & & & & & k_{65}^{(2)} & k_{66}^{(2)} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ u_1^{(3)} \\ u_2^{(3)} \\ u_1^{(4)} \\ u_2^{(4)} \end{bmatrix}$$

In code we add each $[k^{e}]$ to right place in $[K]$

Potential energy of external forces

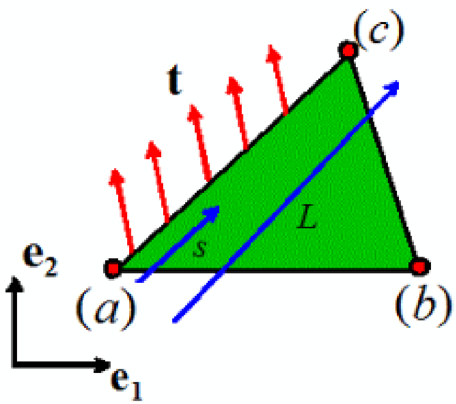


Define $\mathcal{P} = -\int_{S_2} \underline{t}^* \cdot \underline{u} \, ds$.

Consider contribution from 1 element face

Recall $\underline{u}$ varies linearly from $(a) \to (c)$

$$\underline{u} = (1 - s/L)\, \underline{u}^a + (s/L)\, \underline{u}^c$$



Hence

$$\mathcal{P}^{face} = -\int_0^L \underline{u} \cdot \underline{t}^* \, ds$$

$$= -\frac{L}{2} \underline{u}^a \cdot \underline{t}^* - \frac{L}{2} \underline{u}^c \cdot \underline{t}^*$$

Re-write this as $\rho = -\underline{u}^{face\,T}\,\underline{r}^{face}$

$$\underline{u}^{face} = [\,u_1^a \quad u_2^a \quad u_1^c \quad u_2^c\,]$$

$$\underline{r}^{face} = \frac{L}{2}[\,t_1^* \quad t_2^* \quad t_1^* \quad t_2^*\,]$$

Re-write in terms of global displacement

$$\rho^{total} = \sum_{faces} \rho = -\sum_{faces} \underline{u}^{face\,T}\,\underline{r}^{face}$$

Define $\underline{r} = \sum_{faces} \underline{r}^{face}$ (generalized sum - add r^{face} to correct row of $\underline{r}$)

$$\rho^{total} = -\underbrace{\underline{u}}_{2N}{}^{T}\,\underbrace{\underline{r}}_{2N}$$

both 2N vectors

Minimizing the PE

We have $\underline{\Phi} = \frac{1}{2} \underline{u}^T [K] \underline{u} - \underline{u}^T \underline{r}$

Rewrite as $\Phi = \frac{1}{2} u_i K_{ij} u_j - u_i r_i$

At minimum: $\frac{\partial \Phi}{\partial u_k} = 0 \qquad k = 1, \ldots 2N$

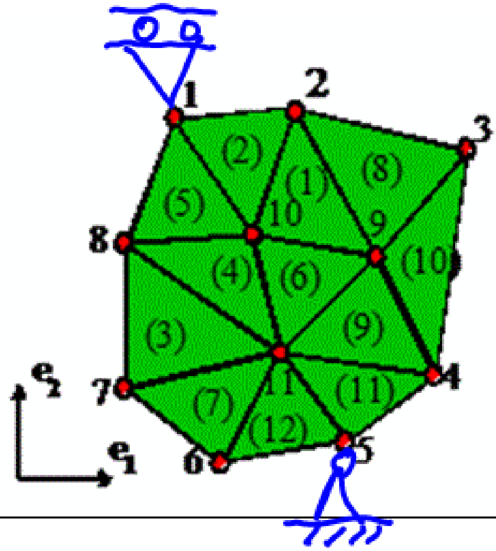
$$\Rightarrow \frac{1}{2} K_{kj} u_j + \frac{1}{2} u_i K_{ik} - r_k = 0$$

$$\Rightarrow K_{kj} u_j - r_k = 0$$

$$[K] \underline{u} = \underline{r}$$

Enforcing Prescribed displacements

Enforce $\underline{u} = \underline{u}^*$ on S_1



Modify equation system to replace equations for forces with equations for displacements

eg to enforce $u_2 = \Delta$ at node 1

Original eqs:

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{12N} \\ k_{21} & k_{22} & & k_{22N} \\ \vdots & & \ddots & \\ k_{2N1} & k_{2N2} & & k_{2N2N} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \vdots \\ u_2^{(N)} \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_4 \end{bmatrix}$$

re-write as

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1\,2N} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ k_{2N\,1} & k_{2N\,2} & & k_{2N\,2N} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \vdots \\ u_2^{(N)} \end{bmatrix} = \begin{bmatrix} r_1 \\ \Delta \\ \vdots \\ r_4 \end{bmatrix}$$

← 2nd row says
$u_2^1 = \Delta$

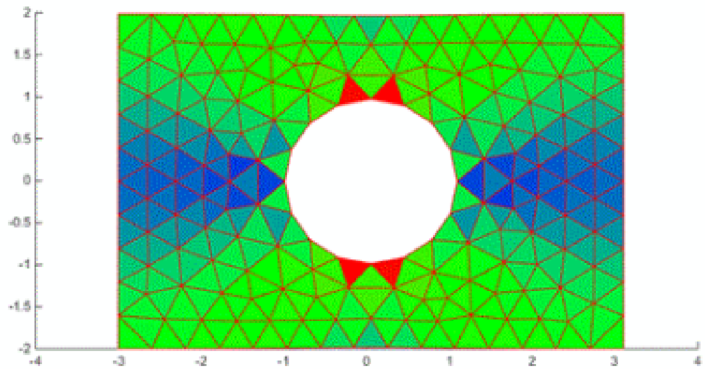
We can symmetrize the system (optional)

Need zeros on 2^{nd} column - we can subtract appropriate multiples of second row from all other rows

$$\begin{bmatrix} k_{11} & 0 & \cdots & k_{1\,2N} \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ k_{2N\,1} & 0 & & k_{2N\,2N} \end{bmatrix} \begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ \vdots \\ u_2^{(N)} \end{bmatrix} = \begin{bmatrix} r_1 - k_{12}\Delta \\ \Delta \\ \vdots \\ r_4 - k_{2N\,2}\Delta \end{bmatrix}$$

Structure of a simple FEA code

- Read data defining problem:
  - Material properties
  - Nodal coordinates
  - Element connectivity
  - List of nodes with prescribed DOF
  - List of elements with loaded faces



- Loop over elements
  - Compute element stiffness, add to global stiffness
- Loop over elements with loaded faces
  - Compute element force vector, add to global force vector
- Modify stiffness and RHS to impose prescribed disps.
- Solve FEA equations for unknown nodal displacements
- Post-processing – compute element strains & stresses