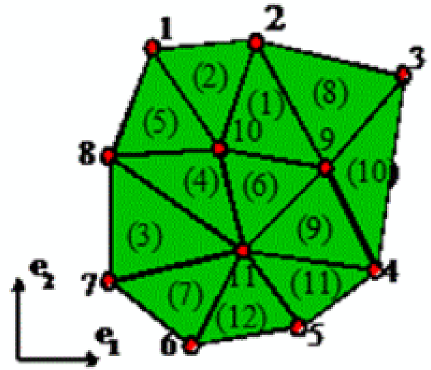
Review – FEA equations from principle of virtual work

- Weak form of equilibrium equation (small strains)

$$\int_R \sigma_{ij} \frac{\partial \delta u_i}{\partial x_j} dV_0 - \int_{S_2} t_i \delta u_i dA = 0 \qquad \forall \; admiss \;\; \delta u_i \Leftrightarrow \partial \sigma_{ij} / \partial x_i = 0 \;\; n_i \sigma_{ij} = t_j^* \;\; on \;\; S_2$$

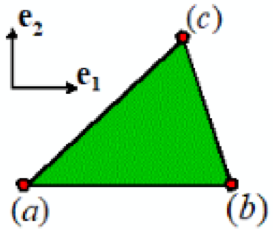


- More general elasticity problem (nonzero initial stress)

$$\sigma_{ij} = \sigma_{ij}^0 + C_{ijkl} \Delta \varepsilon_{kl} = \sigma_{ij}^0 + C_{ijkl} \frac{\partial \Delta u_k}{\partial x_l}$$

- General interpolation scheme

$$\Delta u_i(\mathbf{x}) = \sum_{a=1}^{n} N^a(\mathbf{x}) \Delta u_i^a \qquad \delta u_i(\mathbf{x}) = \sum_{a=1}^{n} N^a(\mathbf{x}) \delta u_i^a$$



- Substitute into PVW

$$\left(K_{aibk} \Delta u_k^b - R_i^a - F_i^a \right) \delta u_i^a = 0 \Rightarrow K_{aibk} \Delta u_k^b = R_i^a + F_i^a$$

ABAQUS uses a - sign

$$K_{aibk} = \int_{\Re} C_{ijkl} \frac{\partial N^a(\mathbf{x})}{\partial x_j} \frac{\partial N^b(\mathbf{x})}{\partial x_l} dV \quad R_i^a = -\int_R \sigma_{ij}^0 \frac{\partial N^a(\mathbf{x})}{\partial x_j} dV \quad F_i^a = \int_R b_i N^a(\mathbf{x}) dV + \int_{S_2} t_i^* N^a(\mathbf{x}) dA$$

ABAQUS uses a - sign

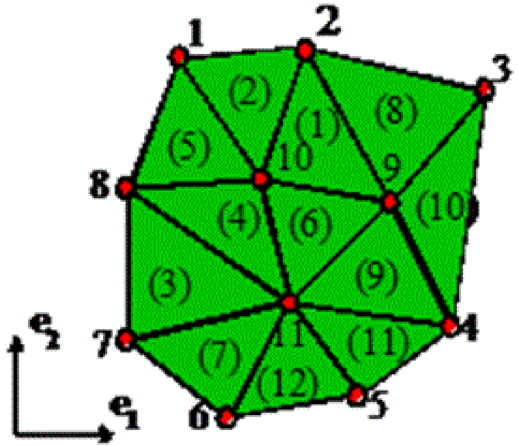
Review – FEA equations from principle of virtual work

$$K_{aibk}$$

Node DOF Node DOF

$$R_{ai}$$

Node DOF



$$[\mathrm{K}] = \begin{bmatrix} K_{1111} & K_{1112} & K_{1113} & K_{1121} & K_{1122} & K_{1123} & \\ K_{1211} & & & & & & \\ K_{1311} & & & & & & \cdots\cdots \\ K_{2111} & & & & & & \\ K_{2211} & & & & & & \\ \vdots & & & & & & \end{bmatrix}$$

$$[R] = \begin{bmatrix} R_{11} \\ R_{12} \\ R_{13} \\ R_{21} \\ R_{22} \\ \vdots \end{bmatrix}$$

Topics for todays class

- Generalizing FEA procedures for elasticity
  - Assembling [K] and [R]
  - Generalized interpolation functions
  - Calculating derivatives of interpolation functions
  - Matrix forms for element stiffness and residual force
  - Calculating volume integrals by numerical integration

Discrete form of PVW

$$\left[K_{aibk}\,\Delta u_k^b - R_{ai} - f_{ai}\right]\delta u_i^a = 0 \qquad \forall \text{ admiss } \delta u_i^a$$

For suitable choice of N^a this requires

$$K_{aibk}\,\Delta u_k^b - R_{ai} - f_{ai} = 0 \qquad \forall\, a \text{ not on } S_1$$

$$\Delta u_k^b = \Delta u_k^*(\underline{x}^b) \qquad \forall\, b \text{ on } S_1$$

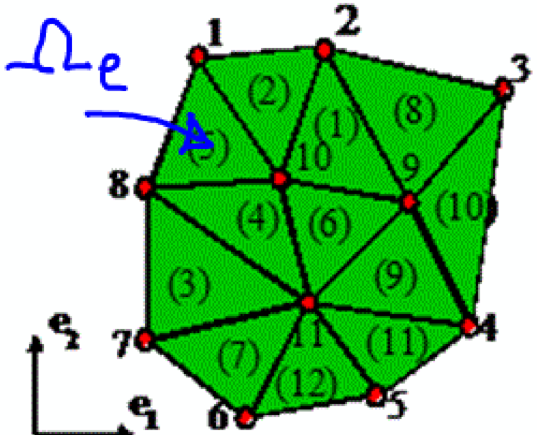
This is a linear system

$$\boxed{[K]\,\Delta \underline{u} = \underline{R} + \underline{f}}$$

4.2 Assembling System

Note $\int_R dV$ can be evaluated element by element

$$\int_R dV = \sum_{\text{elements}} \int_{\Omega_e} dV$$



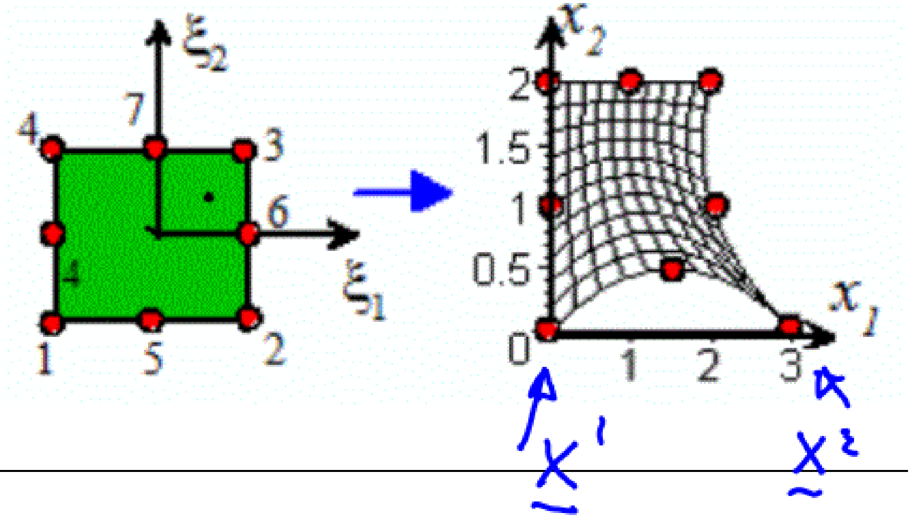
n nodes per el

We can choose $N^a = 0$ for all a not attached to Ω_e

Hence only small parts of k_{aibk} are nonzero. Define

$$\underbrace{k^{el}_{aibk}}_{3n \times 3n \text{ matrix}} = \int_{\Omega_e} C_{ijkl} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial x_j} dV \qquad R_{ai} = -\int_{\Omega_e} \sigma^0_{ij} \frac{\partial N^a}{\partial x_j} dV$$

4.3 Interpolation functions



Not easy to define $N^a(\underset{\sim}{x})$ for arbitrary geometry

Solution: Derive N^a for element with simple geometry and map to actual geometry

Let $-1 < \xi_i < +1$ Define $N^a(\xi_1, \xi_2, \xi_3)$

Then $\underset{\sim}{u} = \underbrace{\underset{\sim}{u}^a}_{3\times n \text{ matrix}} N^a(\xi_i)$ n length vector

$\underset{\sim}{x} = \underset{\sim}{x}^a N^a(\xi_i)$

"Isoparametric elements" – use same N^a for $\underset{\sim}{u}$ and $\underset{\sim}{x}$

As long as mapping $\xi \to \underline{x}$ is invertible this defines $N(\underline{x})$

Computing interpolation function derivatives

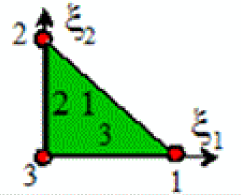
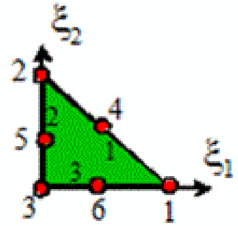
Chain rule $$\frac{\partial u_i}{\partial x_k} = \frac{\partial u_i}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_k} = \underset{3\times n}{u_i^a}\ \underset{n\times 3}{\frac{\partial N^a}{\partial \xi_j}}\ \underbrace{\frac{\partial \xi_j}{\partial x_k}}_{3\times 3}$$

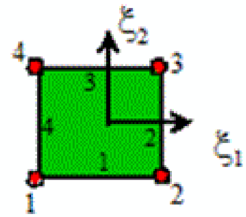
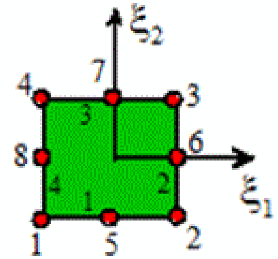
Note $$\underbrace{\frac{d x_i}{\partial \xi_j}}_{3\times 3} = \underbrace{x_i^a}_{3\times n}\ \underbrace{\frac{\partial N^a}{\partial \xi_j}}_{n\times 3}$$

Note $$\left[\frac{d \xi_j}{\partial x_i}\right] = \left[\frac{\partial \underline{x}}{\partial \underline{\xi}}\right]^{-1}_{ji}$$ (inverse of 3x3)

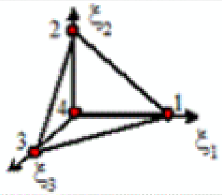
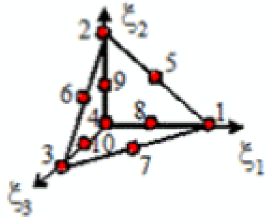
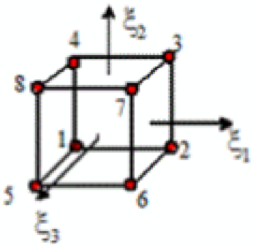
Element interpolations

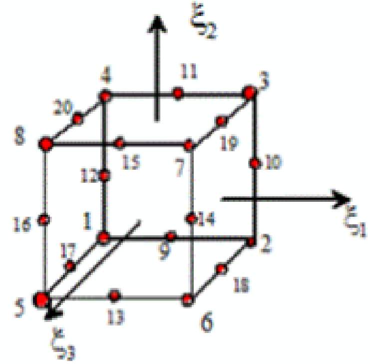
2D elements

$N^1=\xi_1 \quad N^2=\xi_2$ $N^3=1-\xi_1-\xi_2$	
$N^1=(2\xi_1-1)\xi_1 \quad N^2=(2\xi_2-1)\xi_2$ $N^3=(2(1-\xi_1-\xi_2)-1)(1-\xi_1-\xi_2)$ $N^4=4\xi_1\xi_2 \quad N^5=4\xi_2(1-\xi_1-\xi_2)$ $N^6=4\xi_1(1-\xi_1-\xi_2)$	

$N^1=0.25(1-\xi_1)(1-\xi_2)$ $N^2=0.25(1+\xi_1)(1-\xi_2)$ $N^3=0.25(1+\xi_1)(1+\xi_2)$ $N^4=0.25(1-\xi_1)(1+\xi_2)$	
$N^1=-(1-\xi_1)(1-\xi_2)(1+\xi_1+\xi_2)/4$ $N^2=(1+\xi_1)(1-\xi_2)(\xi_1-\xi_2-1)/4$ $N^3=(1+\xi_1)(1+\xi_2)(\xi_1+\xi_2-1)/4$ $N^4=(1-\xi_1)(1+\xi_2)(\xi_2-\xi_1-1)/4$ $N^5=(1-\xi_1^2)(1-\xi_2)/2 \quad N^6=(1+\xi_1)(1-\xi_2^2)/2$ $N^7=(1-\xi_1^2)(1+\xi_2)/2 \quad N^8=(1-\xi_1)(1-\xi_2^2)/2$	

3D elements

$N^1 = \xi_1 \quad N^2 = \xi_2$ $N^3 = \xi_3 \quad N^4 = 1 - \xi_1 - \xi_2 - \xi_3$	
$N^1 = (2\xi_1 - 1)\xi_1 \quad N^2 = (2\xi_2 - 1)\xi_2$ $N^3 = (2\xi_3 - 1)\xi_3 \quad N^4 = (2\xi_4 - 1)\xi_4$ $N^5 = 4\xi_1\xi_2 \quad N^6 = 4\xi_2\xi_3$ $N^7 = 4\xi_3\xi_1 \quad N^8 = 4\xi_4\xi_1$ $\xi_4 = 1 - \xi_1 - \xi_2 - \xi_3$	
$N^1 = (1-\xi_1)(1-\xi_2)(1-\xi_3)/8 \quad N^2 = (1+\xi_1)(1-\xi_2)(1-\xi_3)/8$ $N^3 = (1+\xi_1)(1+\xi_2)(1-\xi_3)/8 \quad N^4 = (1-\xi_1)(1+\xi_2)(1-\xi_3)/8$ $N^5 = (1-\xi_1)(1-\xi_2)(1+\xi_3)/8 \quad N^6 = (1+\xi_1)(1-\xi_2)(1+\xi_3)/8$ $N^7 = (1+\xi_1)(1+\xi_2)(1+\xi_3)/8 \quad N^8 = (1-\xi_1)(1+\xi_2)(1+\xi_3)/8$	

$N^1 = (1-\xi_1)(1-\xi_2)(1-\xi_3)(-\xi_1-\xi_2-\xi_3-2)/8$ $N^2 = (1+\xi_1)(1-\xi_2)(1-\xi_3)(\xi_1-\xi_2-\xi_3-2)/8$ $N^3 = (1+\xi_1)(1+\xi_2)(1-\xi_3)(\xi_1+\xi_2-\xi_3-2)/8$ $N^4 = (1-\xi_1)(1+\xi_2)(1-\xi_3)(-\xi_1+\xi_2-\xi_3-2)/8$ $N^5 = (1-\xi_1)(1-\xi_2)(1+\xi_3)(-\xi_1-\xi_2+\xi_3-2)/8$ $N^6 = (1+\xi_1)(1-\xi_2)(1+\xi_3)(+\xi_1-\xi_2+\xi_3-2)/8$ $N^7 = (1+\xi_1)(1+\xi_2)(1+\xi_3)(+\xi_1+\xi_2+\xi_3-2)/8$ $N^8 = (1-\xi_1)(1+\xi_2)(1+\xi_3)(-\xi_1+\xi_2+\xi_3-2)/8$ $N^9 = (1-\xi_1^2)(1-\xi_2)(1-\xi_3)/4 \quad N^{10} = (1+\xi_1)(1-\xi_2^2)(1-\xi_3)/4$ $N^{11} = (1-\xi_1^2)(1+\xi_2)(1-\xi_3)/4 \quad N^{12} = (1-\xi_1)(1-\xi_2^2)(1-\xi_3)/4$ $N^{13} = (1-\xi_1^2)(1-\xi_2)(1+\xi_3)/4 \quad N^{14} = (1+\xi_1)(1-\xi_2^2)(1+\xi_3)/4$ $N^{15} = (1-\xi_1^2)(1+\xi_2)(1+\xi_3)/4 \quad N^{16} = (1-\xi_1)(1-\xi_2^2)(1+\xi_3)/4$ $N^{17} = (1-\xi_1)(1-\xi_2)(1-\xi_3^2)/4 \quad N^{18} = (1+\xi_1)(1-\xi_2)(1-\xi_3^2)/4$ $N^{19} = (1+\xi_1)(1+\xi_2)(1-\xi_3^2)/4 \quad N^{20} = (1-\xi_1)(1+\xi_2)(1-\xi_3^2)/4$	

4.4 Matrix forms for $[k^{el}]$ and $\underset{\sim}{R}$

Recall PVW has form $\displaystyle\int_R \sigma_{ij}\,\delta\varepsilon_{ij}\,dV - \int_{S_2} t_i\,\delta u_i = 0$

Define stress & strain vectors

Explicit uses $\frac{1}{2}$

$\underset{\sim}{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, \underbrace{2\varepsilon_{13}, 2\varepsilon_{23}}_{\text{Switched in explicit}}]$ (ABAQUS /standard) (3D)

$\underset{\sim}{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}]$

Define $\underset{\sim}{\varepsilon} = [B]\,\underset{\sim}{u}^{el}$ $\qquad \underset{\sim}{\sigma} = [D]\,\underset{\sim}{\varepsilon}$

$\delta\underset{\sim}{\varepsilon} = [B]\,\delta\underset{\sim}{u}^{el}$

Then $\boxed{[k^{el}] = \displaystyle\int_{\Omega el} [B]^T [D][B]\,dV \qquad \underset{\sim}{R} = -\int_{\Omega el} [B]^T \underset{\sim}{\sigma}^{o}\,dV}$

[B] and [D] in 3D (isotropic elasticity)

$$[B]=\begin{bmatrix} \frac{\partial N^1}{\partial x_1} & 0 & 0 & \frac{\partial N^2}{\partial x_1} & 0 & 0 & \\ 0 & \frac{\partial N^1}{\partial x_2} & 0 & 0 & \frac{\partial N^2}{\partial x_2} & 0 & \\ 0 & 0 & \frac{\partial N^1}{\partial x_3} & 0 & 0 & \frac{\partial N^2}{\partial x_3} & \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_1} & 0 & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_2} & 0 & \cdots\cdots \\ \frac{\partial N^1}{\partial x_3} & 0 & \frac{\partial N^1}{\partial x_1} & & & & \\ 0 & \frac{\partial N^1}{\partial x_3} & \frac{\partial N^1}{\partial x_2} & & & & \end{bmatrix}$$

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & & & 0 \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & \frac{1-2\nu}{2} & & \\ & & & & \frac{1-2\nu}{2} & \\ 0 & & & & & \frac{1-2\nu}{2} \end{bmatrix}$$

[B] and [D] for 2D

Plane Strain

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$[B]=\begin{bmatrix} \frac{\partial N^1}{\partial x_1} & 0 & \frac{\partial N^2}{\partial x_1} & 0 & \frac{\partial N^3}{\partial x_1} & 0 & \\ 0 & \frac{\partial N^1}{\partial x_2} & 0 & \frac{\partial N^2}{\partial x_2} & 0 & \frac{\partial N^3}{\partial x_2} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^3}{\partial x_2} & \frac{\partial N^3}{\partial x_1} & \end{bmatrix}$$

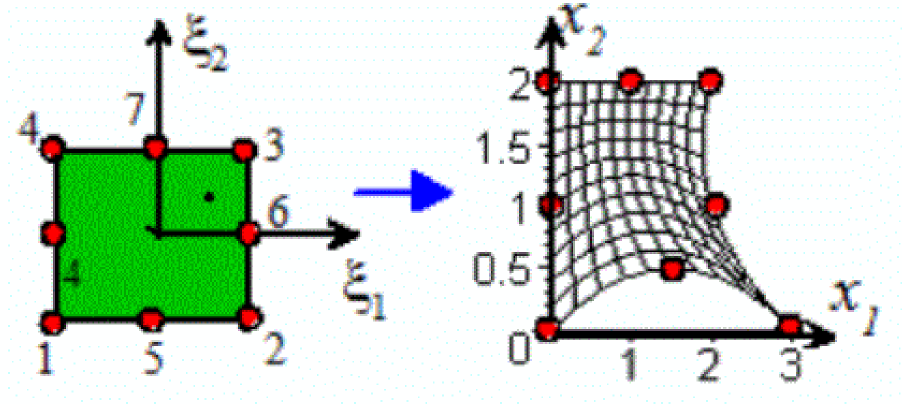
(Plane strain)

Plane Stress

$$[D]=\frac{E}{(1+\nu)(1-\nu)}\begin{bmatrix} 1 & \nu & 0 & 0 \\ \nu & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

4.5 Computing volume integrals

- Now [B] is no longer constant
- Map the integral $\underset{\sim}{x} \rightarrow \underset{\sim}{\xi}$
- Evaluate integrals using Gaussian Quadrature

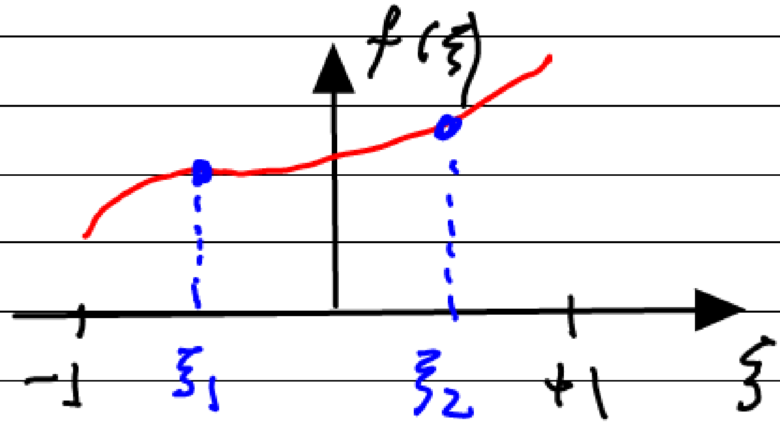


$$\int_{\Omega_{el}} (\,)\, dV = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} (\,)\, \eta \, d\xi_1 \, d\xi_2 \, d\xi_3$$

$$\eta = \det \left[\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\xi}} \right]$$

Gaussian Quadrature (1-D)

$$\int_{-1}^{+1} f(\xi)\, d\xi = \sum_{i=1}^{NINP} W_i\, f(\xi_i)$$



Choose Gauss points ξ_i, W_i to integrate all polynomials up to order $2NINTP - 1$

Example: for $NINTP = 1$

$$\int_{-1}^{+1} d\xi = 2 = W_1 . \Rightarrow W_1 = 2$$

$$\int_{-1}^{+1} \xi\, d\xi = 0 = \xi_1 W_1 \Rightarrow \xi_1 = 0$$

for NINTP = 2

$$\int_{-1}^{+1} d\xi = 2 = W_1 + W_2$$

$$\int_{-1}^{+1} \xi \, d\xi = 0 = \xi_1 W_1 + \xi_2 W_2$$

$$\int_{-1}^{+1} \xi^2 d\xi = \frac{2}{3} = \xi_1^2 W_1 + \xi_2^2 W_2$$

$$\int_{-1}^{+1} \xi^3 d\xi = 0 = \xi_1^3 W_1 + \xi_2^3 W_2$$

Solve for ξ_1, ξ_2, W_1, W_2

$$W_1 = W_2 = 1 \qquad \xi_1 = \frac{-1}{\sqrt{3}} \qquad \xi_2 = \frac{1}{\sqrt{3}}$$

Generalize to 2D or 3D integrals

$$\int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} f(\underline{\xi})\,d\underline{\xi} \approx \sum_{i=1}^{NINTP} w_i\, f(\underline{\xi}_i)$$

Quadrilateral and hexahedral elements

For quadrilateral elements we can simply regard the integral over 2 spatial dimensions as successive 1-D integrals

$$\int_{-1}^{+1}\int_{-1}^{+1} f(\xi_1,\xi_2)\,d\xi_1 d\xi_2 = \sum_{I=1}^{N}\sum_{J=1}^{N} w_I w_J f\left(\xi_1^I,\xi_2^J\right)$$

which gives rise to the following 2D quadrature scheme: Let η_I and v_I for I=1…M denote 1-D quadrature points and weights listed below. Then in 2D, an $N = M \times M$ quadrature scheme can be generated as follows:

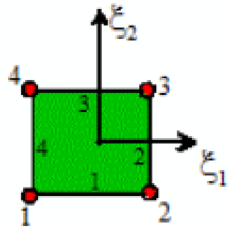
for J=1…M and K=1…M let $\xi_1^I = \eta_J \quad \xi_2^I = \eta_K \quad w_I = v_J v_K, \quad I = M(K-1)+J$

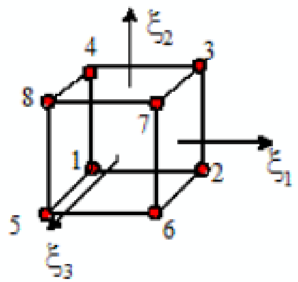
Similarly, in 3D, we generate an $N = M \times M \times M$ scheme as:

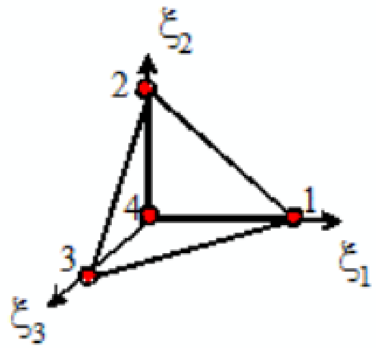
for J=1…M, K=1…M L=1…M let

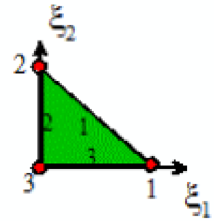
$$\xi_1^I = \eta_J \quad \xi_2^I = \eta_K \quad \xi_3^I = \eta_L \quad w_I = v_J v_K v_L, \quad I = M^2(L-1)+M(K-1)+J$$

M=1	$\eta_1 = 0$	$v_1 = 2$
M=2	$\eta_1 = -0.5773502691$ $\eta_2 = 0.5773502691$	$v_1 = 1.0$ $v_2 = 1.0$
M=3	$\eta_1 = -0.7745966692$ $\eta_2 = 0.$ $\eta_3 = 0.7745966692$	$v_1 = 0.55555555555$ $v_2 = 0.88888888888$ $v_3 = 0.55555555555$





Integration points for tetrahedral elements		
1 point	$\xi_1^1 = 1/4 \quad \xi_2^1 = 1/4 \quad \xi_3^1 = 1/4 \quad w_1 = 1/6$	
4 point	$\xi_1^1 = \alpha \quad \xi_2^1 = \beta \quad \xi_3^1 = \beta \quad w_1 = 1/24$ $\xi_1^2 = \beta \quad \xi_2^1 = \alpha \quad \xi_3^2 = \beta \quad w_2 = 1/24$ $\xi_1^3 = \beta \quad \xi_2^3 = \beta \quad \xi_3^3 = \alpha \quad w_3 = 1/24$ $\xi_1^4 = \beta \quad \xi_2^4 = \beta \quad \xi_3^4 = \beta \quad w_4 = 1/24$ where $\alpha = 0.58541020, \quad \beta = 0.13819660$	

Integration points for triangular elements		
1 point	$\xi_1^1 = 1/3 \quad \xi_2^1 = 1/3 \quad w_1 = 1/2$	
3 point	$\xi_1^1 = 1/2 \quad \xi_2^1 = 0 \quad w_1 = 1/6 \qquad \xi_1^1 = 0.6 \quad \xi_2^1 = 0.2 \quad w_1 = 1/6$ $\xi_1^2 = 0 \quad \xi_2^2 = 1/2 \quad w_2 = 1/6$ or $\xi_1^2 = 0.2 \quad \xi_2^2 = 0.6 \quad w_2 = 1/6$ $\xi_1^3 = 1/2 \quad \xi_2^3 = 1/2 \quad w_3 = 1/6 \qquad \xi_1^3 = 0.2 \quad \xi_2^3 = 0.2 \quad w_3 = 1/6$ (the first scheme here is optimal, but has some disadvantages for quadratic elements because the integration points coincide with the midside nodes. The second scheme is less accurate but more robust).	
4 point	$\xi_1^1 = 0.6 \quad \xi_2^1 = 0.2 \quad w_1 = 25/96$ $\xi_1^2 = 0.2 \quad \xi_2^2 = 0.6 \quad w_2 = 25/96$ $\xi_1^3 = 0.2 \quad \xi_2^3 = 0.2 \quad w_3 = 25/96$ $\xi_1^4 = 1/3 \quad \xi_2^4 = 1/3 \quad w_4 = -27/96$	

How many integration points are needed?
We can find order of polynomials in each N^a

Number of integration points for fully integrated elements	
Linear triangle (3 nodes): 1 point	Linear tetrahedron (4 nodes): 1 point
Quadratic triangle (6 nodes): 4 points	Quadratic tetrahedron (10 nodes): 4 points
Linear quadrilateral (4 nodes): 4 points	Linear brick (8 nodes): 8 points
Quadratic quadrilateral (8 nodes): 9 points	Quadratic brick (20 nodes): 27 points

final formulas for $[k^{el}]$, $\underline{R}$

$$[k^{el}] = \sum_{i=1}^{NINTP} W_i [B(\xi_i)]^T [D] [B(\xi_i)] \eta(\xi_i)$$

$$\underline{R} = -\sum_{i=1}^{NINTP} W_i [B(\xi_i)]^T \underline{\sigma}^{\circ} \eta(\xi_i)$$