Review: FEA as a Galerkin Method

- Strong form of equilibrium equation (small strains)
  \[ \frac{\partial}{\partial x_j} \left( C_{ijkl} \frac{\partial u_k}{\partial x_l} \right) + b_i = 0 \]
  \[ t_i^* - C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j = 0 \]

- Introduce test function \( \eta_i \) satisfying \( \eta_i = 0 \) on \( S_1 \) multiply strong form and integrate
  \[ \int_R \frac{\partial}{\partial x_j} \left( C_{ijkl} \frac{\partial u_k}{\partial x_l} \right) \eta_i dV_0 + \int_R b_i \eta_i dV_0 + \int_{\bar{R}} \left( t_i^* - C_{ijkl} \frac{\partial u_k}{\partial x_l} n_j \right) dA = 0 \quad \forall \text{admiss} \eta_i \]

- Integrate first term by parts
  \[ -\int_R C_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \eta_i}{\partial x_j} dV_0 + \int_{\bar{R}} b_i \eta_i dV_0 + \int_{S_2} t_i^* \eta_i dA = 0 \quad \forall \text{admiss} \eta_i \]

- Insert interpolation
  \[ u_i(x) = \sum_{a=1}^{n} N^a(x) \delta u^a_i \]
  \[ \eta_i(x) = \sum_{a=1}^{n} N^a(x) \eta^a_i \]

  \[ \left( K_{\text{cijkl}} u^b_k - F_i^a \right) \eta^a_i = 0 \Rightarrow K_{\text{cijkl}} u^b_k = F_i^a \]

  \[ K_{\text{cijkl}} = \int_R C_{ijkl} \frac{\partial N^a(x)}{\partial x_j} \frac{\partial N^b(x)}{\partial x_l} dV \quad F_i^a = \int_R b_i N^a(x) dV + \int_{S_2} t_i^* N^a(x) dA \]
Topics for today's class

- Perspectives on FEA for linear elasticity
  - FEA as a best approximation method
  - Conditions for FEA convergence
- Problems with standard elements
  - 'Shear Locking' in beam/plate/shell problems
  - 'Volumetric locking' in near-incompressible materials
5.2 FEA as a “best approximation” method

Linear Elasticity, elastic constants $C_{ijk}\ell$, $t_i^*$ prescribed on $S_i$

Let $u$ be exact solution 

$$u^h = \sum a_i^q u^q$$ be the FEA solution

Define $E(u - u^h)$ as error measure

$$E(u - u^h) = \int_\Omega \frac{1}{2} C_{ijk\ell} \frac{\partial}{\partial x_k} (u^k - u_h^k) \frac{\partial}{\partial x_j} (u^j - u_h^j) \, dV$$

Strain energy of $u - u^h$

Then FEA minimizes $E$ w.r.t $u^q$
Proof: Re-write $\varepsilon$ as

$$\varepsilon(n) + \varepsilon(y^n) = \int_{\mathbb{R}} \left( \frac{1}{2} (\xi_{ij} + \xi_{ji}) \frac{\partial u_i}{\partial x_j} \frac{\partial u_k}{\partial x_i} \right) dx$$

Note: $\xi_{ij} = \xi_{ji}$

$$\int_{\mathbb{R}} \xi_{ij} \frac{\partial u_i}{\partial x_j} = \int_{\mathbb{R}} \frac{\partial}{\partial x_j} \xi_{ij} u^k - \frac{\partial}{\partial x_j} \xi_{ij} u_i$$

$$= \int_{\mathbb{R}} \frac{\partial}{\partial x_j} \xi_{ij} u_i \eta_j dA = \int_{S^2} e_i u^h dA + \int_{S_1} \int_{S_2} \xi_{ij} \eta_j u_i \eta_j dA$$
\[ E(u - u^h) = E(u) + E(u^h) - \int_{s_2} t_i u_i^h dA - \int_{s_1} \sigma_{ij} n_j u_i^h dA \]

Potential energy of FEA solution

Now minimize \( E \) w.r.t. \( u^h \), with fixed \( u^h = u^* \) on \( s_1 \)

\[ \Rightarrow \text{Minimize PE of } u^h \]

\[ \Rightarrow \text{FEA equations follow!} \]
5.3 Conditions for convergence of FEA for linear elasticity

State results w/o proof: see Bathe chap 4 for derivations

Factors affecting convergence

1. \( m \): Order of highest derivative appearing in weak form of PDE

Examples 2D/3D linear elasticity

\[ \int C_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial u_l}{\partial x_j} \, dV - \int \epsilon_{ij} \eta_i \, dA \Rightarrow m = 1 \]

Beam:

\[ \int_0^L \left( EI \frac{d^2 w}{dx^2} \frac{d^2 w}{dx^2} - q \eta \right) \, dx \Rightarrow m = 2 \]
Order of highest complete polynomial in interpolation functions

$k$ is often displayed as Pascal triangle for 2D elements

$k = 1$ for linear

$k = 2$ for quadratic
Element size and shape

Let he be smallest sphere enclosing element pe " largest sphere that fits in element

Define \( h = \max \{ h_e \} \quad p = \min \{ p_e \} \)

Let \( \sigma = \frac{h}{p} \)

We can show that

\[
E = B(U) \frac{h^{k+1}}{p^m} = B(U) \sigma^m h^{k-m+1}
\]

We need \( E \to 0 \) as \( h \to 0 \)
This requires:

1. \( \mathcal{E}(U^h) \) must be bounded
2. \( k - m + 1 > 0 \)

Consequences:

1 \( \Rightarrow \) for linear elasticity we need \( C^1 \) continuity in \( \mathbb{R} \); at linear interpolation in \( \mathbb{R}^e \)

\[ C^0 \Rightarrow \text{No good} \quad \quad \quad \quad c' : \text{OK} \]
For beams: at least quadratic interpolation
Continuous slope across element boundaries

2nd condition \[ k - m + 1 > 0 \]

For linear elasticity \( m = 1 \Rightarrow k > 0 \) (at least linear)

For beam: \( m = 2 \Rightarrow k \geq 1 \); at least quadratic.
5.4 "Patch Test" - an engineering test of a solid mechanic element

Although passing patch test does not guarantee convergence, usually elements that pass are OK, others are not.

Two version:

(1): Generate meshes (test several) of irregular elements

Apply uniform strain field to mesh by constraining nodal displacements

\[ U_i^q = E_{ij} x_j^q \]  

↑ constant strain
Patch test passed if $k u = \varnothing$

for all interior nodes

(2) Apply (1) to all exterior nodes

Solve $[K]u = f$

At internal nodes we should get

$u_i^a = E_{ij} x_j^a$
6) **Advanced element formulations**

Standard elements fail in two situations:

1. "Shear locking" in beams
2. "Volumetric locking" in near incompressible materials

6.1 **Shear locking**

Analytical solution to 2D elasticity problem

(See Chapter 5 of solidmechanics.org for details)

\[
\begin{align*}
u_1 &= \frac{3P}{4Ea^3b} x_1 x_2 - \frac{P}{4Ea^3b} (2 + \nu) x_1^3 + \frac{3P}{2Ea^3b} (1 + \nu) a^2 x_2 - \frac{3PL^2 x_2}{4Ea^3b} \\
u_2 &= -\nu \frac{3P}{4Ea^3b} x_1 x_2 - \frac{P}{4Ea^3b} x_1^3 + \frac{3PL^2 x_1}{4Ea^3b} - \frac{PL^3}{2Ea^3b}
\end{align*}
\]
Try to solve this with FEA

1. Solve a short beam with linear elements
   ⇒ OK

2. Long beam with linear elements
   FEA underestimates solution by factor of 10

3. Quadratic 8-noded elements are OK