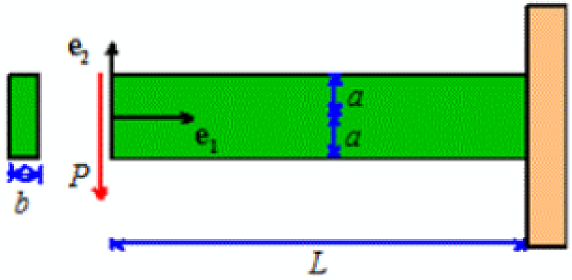
Review: Shear Locking

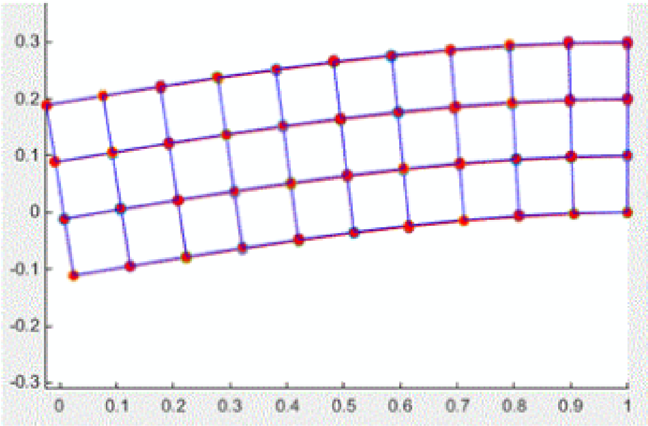
Analytical solution to 2D elasticity problem
(See Chapter 5 of solidmechanics.org for details)

$$u_1 = \frac{3P}{4Ea^3b}x_1^2x_2 - \frac{P}{4Ea^3b}(2+\nu)x_2^3 + \frac{3P}{2Ea^3b}(1+\nu)a^2x_2 - \frac{3PL^2x_2}{4Ea^3b}$$

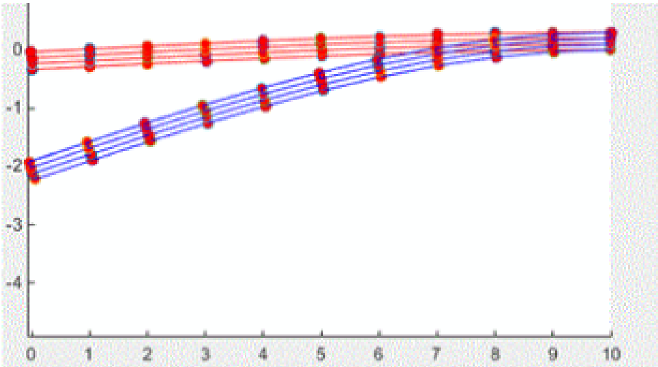
$$u_2 = -\nu\frac{3P}{4Ea^3b}x_1x_2^2 - \frac{P}{4Ea^3b}x_1^3 + \frac{3PL^2x_1}{4Ea^3b} - \frac{PL^3}{2Ea^3b}$$



Solution for short beam is OK (elements are square)



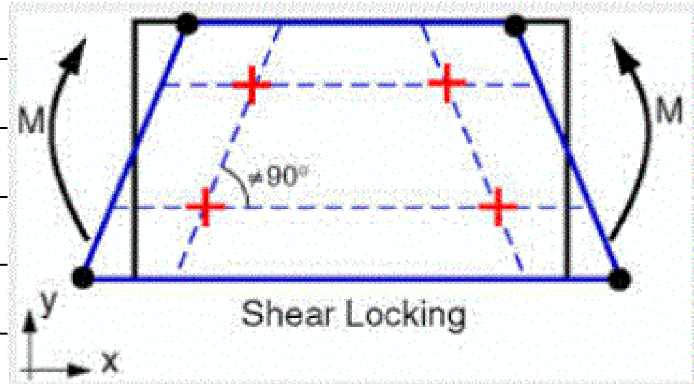
Linear elements underestimate solution by large factor
Caused by “Shear Locking”



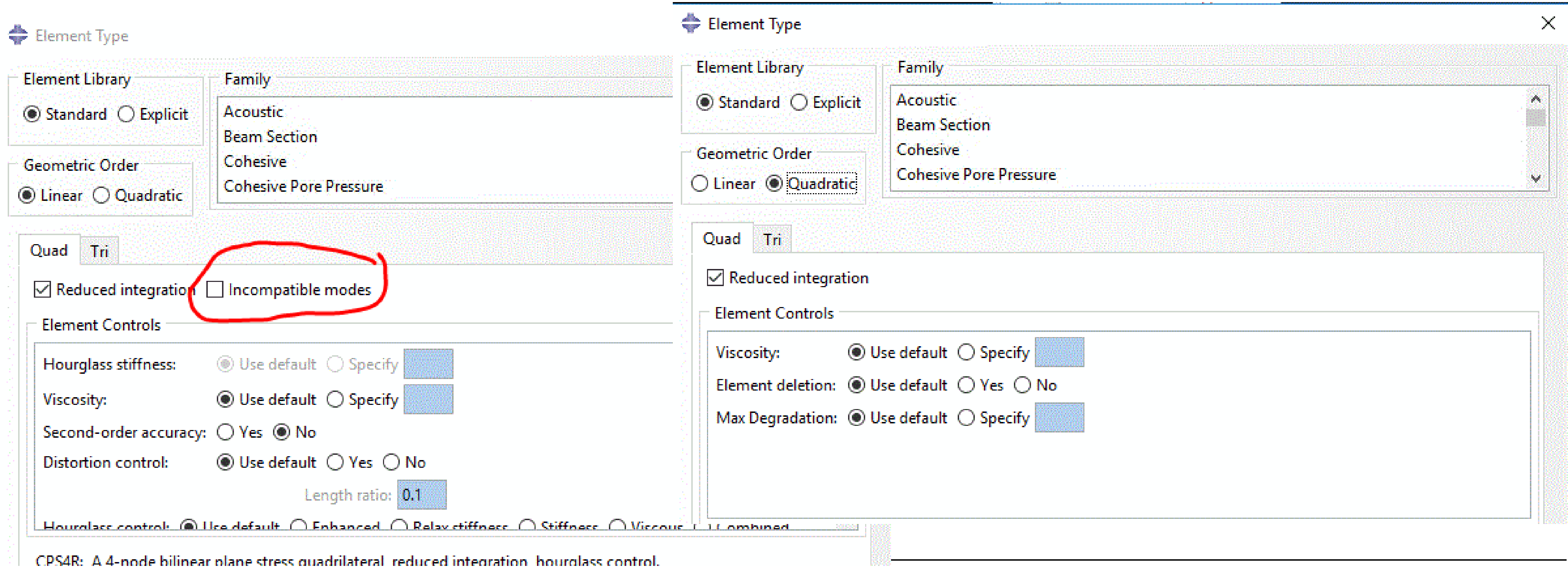
Topics for todays class

- Problems with standard elements
  - 'Shear Locking' in beam/plate/shell problems
  - 'Volumetric locking' in near-incompressible materials

- Standard linear elements cannot interpolate bending fields
- Bending linear elements induces a large spurious shear strain at integration points
  - Spuriously stiff beam
  - Fix with "Incompatible mode" elements



• ABAQUS :

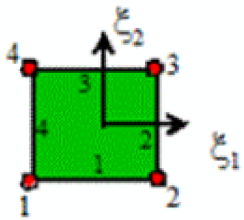

Element Type
Element Library
Standard
Explicit
Family
Acoustic
Beam Section
Cohesive
Cohesive Pore Pressure
Geometric Order
Linear
Quadratic
Quad
Tri
Reduced integration
Incompatible modes
Element Controls
Hourglass stiffness:
Use default
Specify
Viscosity:
Use default
Specify
Second-order accuracy:
Yes
No
Distortion control:
Use default
Yes
No
Length ratio:
0.1
CPS4R: A 4-node bilinear plane stress quadrilateral, reduced integration, hourglass control.
Element Type
Element Library
Standard
Explicit
Family
Acoustic
Beam Section
Cohesive
Cohesive Pore Pressure
Geometric Order
Linear
Quadratic
Quad
Tri
Reduced integration
Element Controls
Viscosity:
Use default
Specify
Element deletion:
Use default
Yes
No
Max Degradation:
Use default
Specify

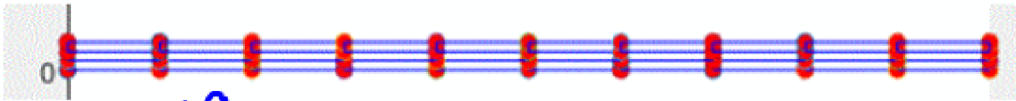
Fix is to introduce additional DOF to interpolate strains inside element

Describe displacement with standard isoparametric mapping

$$\underline{u} = N^a(\underline{\xi})\, \underline{u}^a \qquad \underline{x} = N^a(\underline{\xi})\, \underline{x}^a$$

$$N^1 = 0.25(1-\xi_1)(1-\xi_2)$$
$$N^2 = 0.25(1+\xi_1)(1-\xi_2)$$
$$N^3 = 0.25(1+\xi_1)(1+\xi_2)$$
$$N^4 = 0.25(1-\xi_1)(1+\xi_2)$$





$\underline{x}^a$

Modify displacement gradient

$$\frac{\partial u_i}{\partial x_j} = \left\{ \frac{\partial N^a}{\partial \xi_m} u_i^{a} + \sum_{k=1}^{p} \frac{\eta(0)}{\eta(\underline{\xi})} \underbrace{\alpha_i^{(k)}}_{\text{new element DOFs}} \delta_{km}\, \xi_k \right\} \frac{\partial \xi_m}{\partial x_j}$$

$p = \#$ coords $\qquad \eta = \det\left(\dfrac{d\underline{x}}{d\underline{\xi}}\right)$

Interpolate virtual displacement δu_i and gradient the same way:

$$\frac{\partial \delta u_i}{\partial x_j} = \left\{ \frac{\partial N^a}{\partial \xi_m} \delta U_i^a + \sum_{k=1}^{p} \frac{\eta(0)}{\eta(\xi)} \delta \alpha_i^{(k)} \delta_{km} \xi_k \right\} \frac{\partial \xi_m}{\partial x_j}$$

PVW:

$$\int_R C_{ijkl} \frac{\partial u_k}{\partial x_l} \frac{\partial \delta u_i}{\partial x_j} dV + \int_R \sigma_{ij}^0 \frac{\partial \delta u_i}{\partial x_j} dV - \int_{S_2} t_i^* \delta u_i \, dA = 0$$

New discrete system has form

$$\left[K^{uu}_{aibk} U_k^b + K^{u\alpha}_{aikl} \alpha_l^{(k)} - R^u_{ai} - f_{ia} \right] \delta U_i^a = 0$$

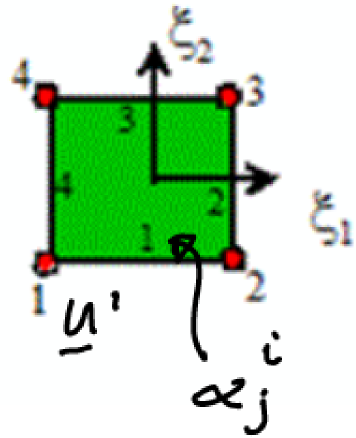
$$\left[K^{\alpha u}_{ijbk} U_k^b + K^{\alpha\alpha}_{ijkl} \alpha_l^k - R^\alpha_{ij} \right] \delta \alpha_j^{(i)} = 0$$

Must hold $\forall$ admiss δU_i^a and $\delta \alpha_j^i$

$\Rightarrow$ New system of FEA eqs

Implementing as code

Focus on element level operations
(Illustrate in 2D)



Define new B matrix

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = [B] \begin{bmatrix} \underline{u} \\ \\ \underline{\alpha} \end{bmatrix}$$

$$\underline{u} = [u_1^1 \; u_2^1 \; \ldots \; u_1^4 \; u_2^4]$$

$$\underline{\alpha} = [\alpha_1^1 \; \alpha_2^1 \; \alpha_1^2 \; \alpha_2^2]$$

$$[B] = \begin{bmatrix} \frac{\partial N^1}{\partial x_1} & 0 & \frac{\partial N^2}{\partial x_1} & 0 & \frac{\partial N^3}{\partial x_1} & 0 & & \frac{\eta(0)}{\eta(\xi)}\xi_1\frac{\partial \xi_1}{\partial x_1} & 0 & \frac{\eta(0)}{\eta(\xi)}\xi_2\frac{\partial \xi_2}{\partial x_1} & 0 \\ 0 & \frac{\partial N^1}{\partial x_2} & 0 & \frac{\partial N^2}{\partial x_2} & 0 & \frac{\partial N^3}{\partial x_2} & \cdots\cdots & 0 & \frac{\eta(0)}{\eta(\xi)}\xi_1\frac{\partial \xi_1}{\partial x_2} & 0 & \frac{\eta(0)}{\eta(\xi)}\xi_2\frac{\partial \xi_2}{\partial x_2} \\ \frac{\partial N^1}{\partial x_2} & \frac{\partial N^1}{\partial x_1} & \frac{\partial N^2}{\partial x_2} & \frac{\partial N^2}{\partial x_1} & \frac{\partial N^3}{\partial x_2} & \frac{\partial N^3}{\partial x_1} & & \frac{\eta(0)}{\eta(\xi)}\xi_1\frac{\partial \xi_1}{\partial x_2} & \frac{\eta(0)}{\eta(\xi)}\xi_1\frac{\partial \xi_1}{\partial x_1} & \frac{\eta(0)}{\eta(\xi)}\xi_2\frac{\partial \xi_2}{\partial x_2} & \frac{\eta(0)}{\eta(\xi)}\xi_2\frac{\partial \xi_2}{\partial x_1} \end{bmatrix}$$

Now define

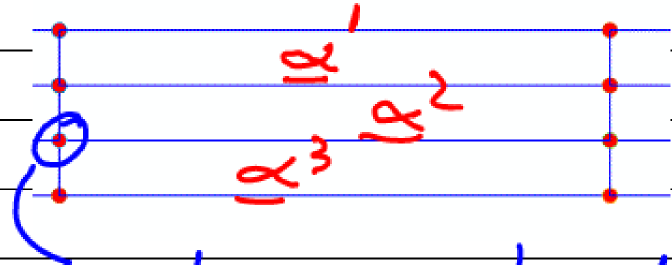
$$\underbrace{[\hat{k}]}_{12\times 12} = \int_{\Omega_e} [B]^T [D][B]\, dV \qquad \underline{R} = -\int_{\Omega_e} [B]^T \underline{\sigma}^0 dV$$

$\hat{k}$ has the following structure;

$$\begin{bmatrix} \overset{8\times 8}{k^{uu}} & \overset{8\times 4}{k^{u\alpha}} \\ \underbrace{k^{\alpha u}}_{4\times 8} & \underbrace{k^{\alpha\alpha}}_{4\times 4} \end{bmatrix} \begin{bmatrix} \underline{u} \\ \underline{\alpha} \end{bmatrix} = \begin{bmatrix} \underline{r}^u \\ \underline{r}^\alpha \end{bmatrix}$$

Notice that $\underset{\sim}{\alpha}$ variables are local to each element

This means we can eliminate $\underset{\sim}{\alpha}$ inside element



nodes are shared across elements

- no need to add to global system

Elimination : $\underset{\sim}{\alpha} = K^{\alpha\alpha -1} \{ \underset{\sim}{r}^{\alpha} - K^{\alpha u} \underset{\sim}{u} \}$ (last 4 rows)

Subst back

$$\underbrace{\{ K^{uu} - K^{u\alpha} K^{\alpha\alpha -1} K^{\alpha u} \}}_{[K^{el}] \quad 8\times 8} \underset{\sim}{u} = \underbrace{\underset{\sim}{r}^{u} - K^{\alpha\alpha -1} \underset{\sim}{r}^{\alpha}}_{8\times 1 \text{ vector } \underset{\sim}{r}^{el}}$$

Summary:

① Assemble $[\hat{K}]$ - use standard code, loop over integration points, etc

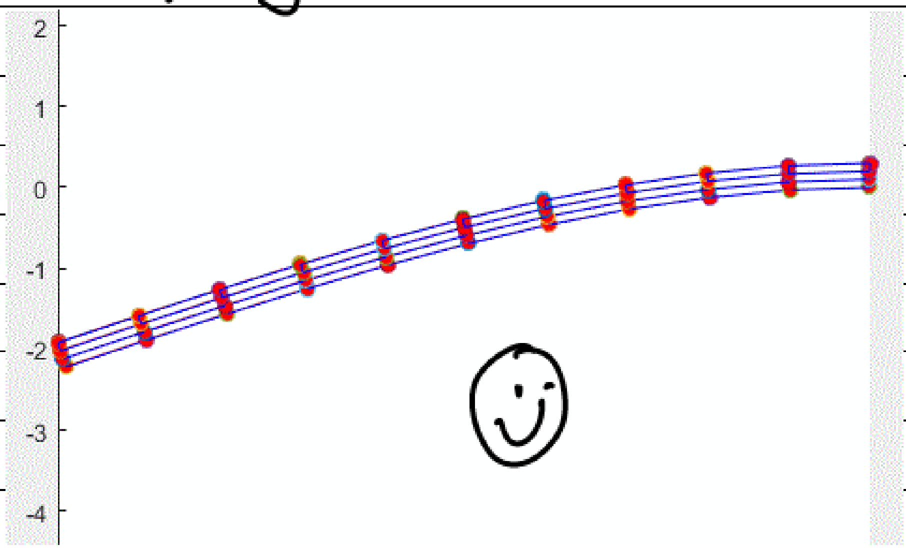
② Extract K^{uu} $K^{u\alpha}$ $K^{\alpha u}$ $K^{\alpha\alpha}$

③ Assemble K^{el}

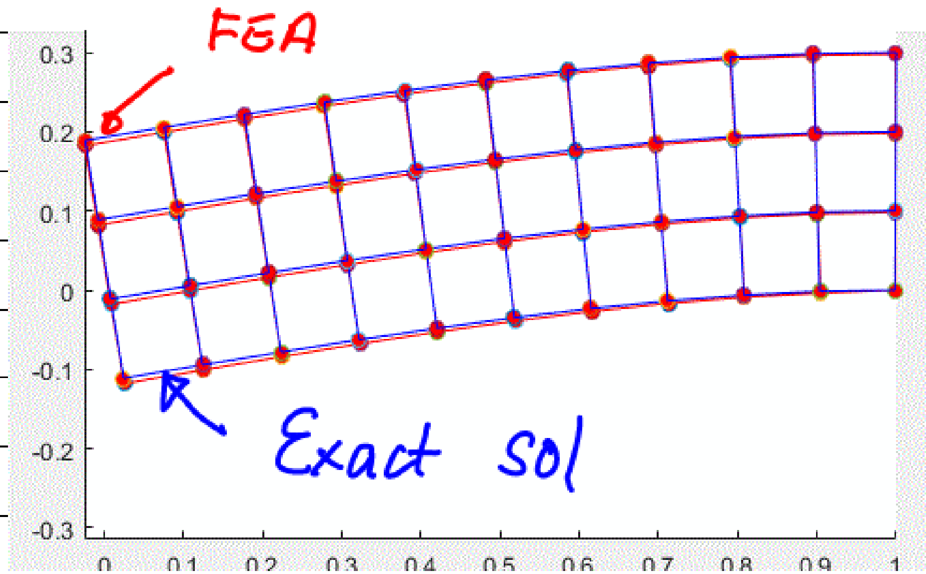
Remainder of code is unaffected

Test beam problem again

Long beam



Short beam.



This can be generalized to construct a family of "assumed strain" elements

Construct with care : Added strain DOF must be independent

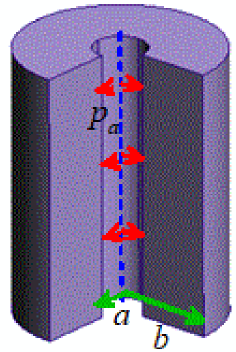
Can perform poorly for finite deformation

6.2) Volumetric Locking in near-incompressible materials

Solve pressurized cylinder problem

Analytical solution to 2D elasticity problem
(See Chapter 4 of solidmechanics.org for details)

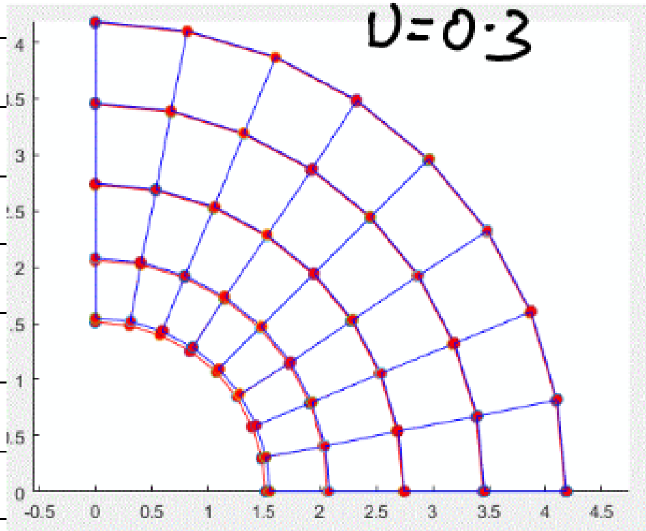
$$\mathbf{u} = \frac{(1+\nu)a^2b^2}{E\left(b^2-a^2\right)}\left\{\frac{p}{r} + (1-2\nu)\frac{p}{b^2}r\right\}\mathbf{e}_r$$

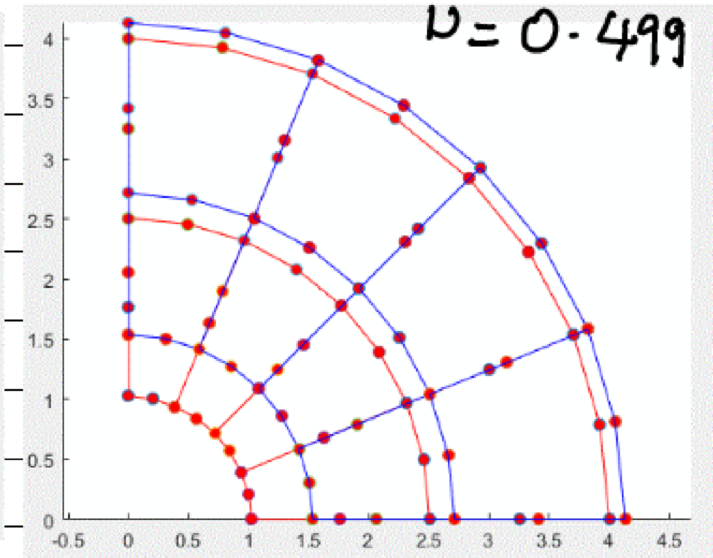


Use fully integrated 2D plane strain elements

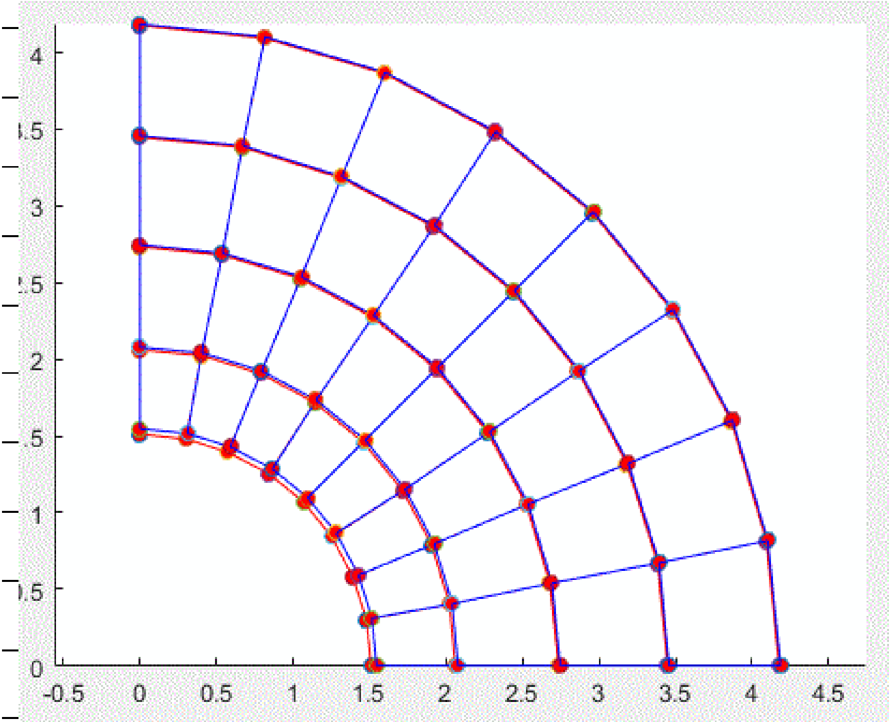
Good accuracy for $\nu = 0.3$ with all element types

For ν close to $\frac{1}{2}$ FEA displacements too small in all element types





This is produced by "volumetric locking"



We need to satisfy $\varepsilon_{kk} \approx 0$ at every integration point

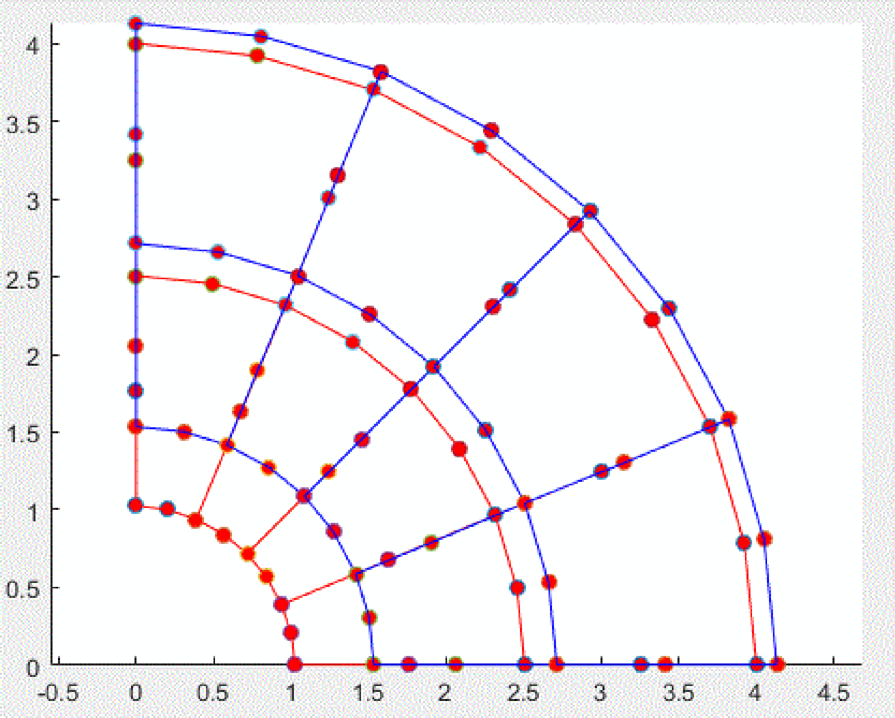
4 int pts per element for linear quads $\Rightarrow$ 128 constraints

80 DOF (# nodes × # DOF − # constrained DOF)

Far more constraints than DOF

Only solution with $\varepsilon_{kk} \approx 0$ is $\underline{u} \approx \underline{0}$

Quadratic



64 DOF

9 int pts per element

8x9 constraints

- same problem !

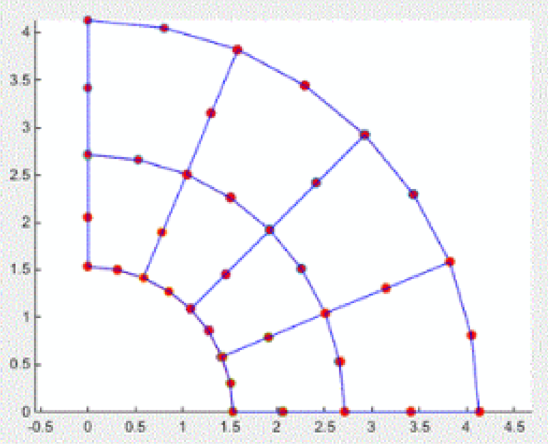
Fixes:

(1) Reduced Integration
(2) "Selective" reduced integration
(3) "B-bar" method
(4) Hybrid, or mixed elements

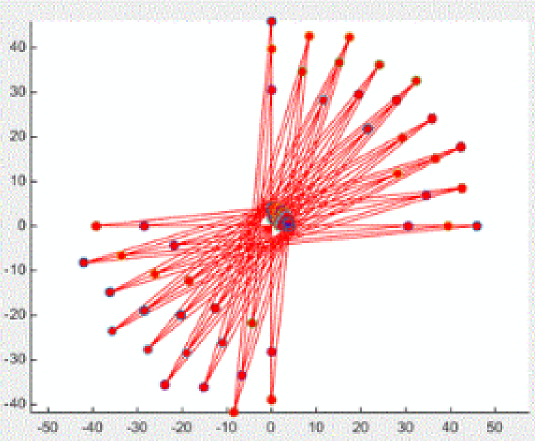
Fix #1 " Reduced Integration " elements

- try reducing # int pts (4 for quadratic
  1 for linear)

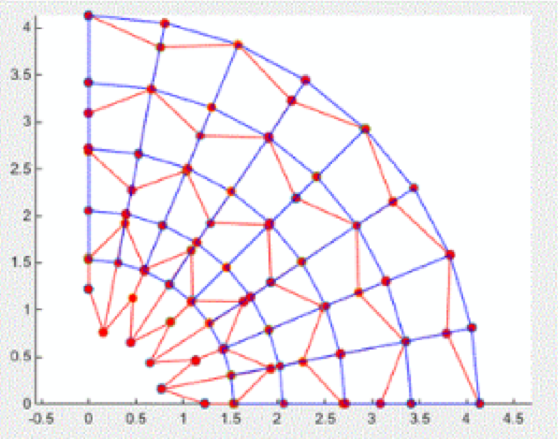
Number of integration points for reduced integration schemes	
Linear triangle (3 nodes) 1 point	Linear tetrahedron (4 nodes): 1 point
Quadratic triangle (6 nodes): 3 points	Quadratic tetrahedron (10 nodes): 4 points
Linear quadrilateral (4 nodes): 1 point	Linear brick (8 nodes): 1 point
Quadratic quadrilateral (8 nodes): 4 points	Quadratic brick (20 nodes): 8 points



8 noded quads –
perfect!



4 noded quads –
Hourglassing



Scaled to show
hourglass mode