The influence of serrated flow on necking in tensile specimens

F. Zhang 1, A.F. Bower *, W.A. Curtin

School of Engineering, Brown University, Providence, RI 02912, USA

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Abstract

Three-dimensional finite element simulations are used to investigate the role of serrated flow on the strain at the onset of necking in a cylindrical uniaxial tension specimen. The material is idealized using a modified form of the McCormick constitutive equation, which has an additional material parameter that allows the rate of transient aging to be varied without affecting its steady-state response. Stability calculations and direct simulations show that, if the transient response is sufficiently slow, serrated flow can be suppressed, even though the material has negative steady-state strain rate sensitivity. This result is then used to determine the effect of suppressing serrated flow on the strain to localization. We find that negative steady-state sensitivity significantly reduces the strain required to initiate necking failure in a tensile specimen. However, the strain to failure is largely unaffected by the transient response of the material, and suppressing the serrated flow in particular has a negligible effect on the localization strain. We conclude that, while both serrated flow and reduced ductility are observed in materials with negative rate sensitivity, the reduction in ductility is not a direct consequence of serrated flow.

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1. Introduction

Portevin-Le Chatelier (PLC) bands are a form of material instability observed in solute strengthened alloys [1]. They consist of narrow bands of intense plastic shear that nucleate and propagate through an otherwise homogeneously deforming specimen. The phenomenon is typically observed over a characteristic range of strain rate and temperature which depends on the composition of the alloy. The bands induce serrations on the stress–strain curve, so this mode of deformation is often referred to as “serrated” or “jerky” flow. Serrated flow is a consequence of a decrease in the steady-state flow stress with increasing strain rate; a phenomenon known as negative strain rate sensitivity. In the same regime of strain rate and temperature, the material also experiences a substantial decrease in the ductility (which is manifested, for example, as a decreased strain to localization in a tensile test). There is thus considerable interest in understanding the underlying causes of negative strain rate sensitivity and in developing strategies to mitigate both serrated flow and low ductility. Although PLC instabilities and reduced ductility occur over similar ranges of temperature and strain rate, the precise role played by the PLC bands in reducing ductility is not well understood. Experiments show that serrated flow can persist over a long period of time prior to necking or fracture, suggesting that PLC bands are not directly responsible for failure [2]. Detailed in situ observations of necking in Al-Mg sheets reported by Kang et al. [3] suggest that, while PLC bands do not in themselves cause failure, they may reduce ductility by increasing the local strain in the diffuse neck and shear bands that develop prior to failure. A sophisticated numerical study of the influence of PLC bands on ductility has been carried out by Hopperstad et al. [4]. Their model accounts for phenomena such as yield surface anisotropy and fracture, in addition to the negative strain rate sensitivity associated with dynamic strain aging. Their simulations predicted a substantial loss
of ductility under conditions where PLC bands were observed to form, and indicated that necking instability, rather than fracture, was the failure mode. An important conclusion of this study was that, although PLC bands do not directly cause failure, they roughen the surface of the specimen, which may initiate a neck.

It is well known that negative steady-state strain rate sensitivity is a necessary, but not sufficient, condition for a material to exhibit serrated flow \cite{5-7}. In particular, a recent observation by Zhang et al. \cite{8} suggests a novel way to examine the connection between PLC bands and ductility. Zhang et al. implemented a kinetic model of dynamic strain aging developed by Soare and Curtin \cite{9,10} as a constitutive law for finite element computations. Although their constitutive law predicted steady-state flow stresses that were in excellent agreement with experiment (including the range of temperature and strain rates that lead to negative strain rate sensitivity), no PLC bands were observed in the finite element computations, even in the regime of negative strain rate sensitivity. Zhang et al. speculated that PLC bands may have been suppressed because the Soare-Curtin model exhibits a very slow transient response to step changes in strain rate. This result is of interest here because it suggests an approach to suppress PLC instabilities, without affecting either the negative steady-state strain rate sensitivity of a material or its hardening behavior. With this background, our objective here is to undertake a more complete numerical study of the dependence of ductility on negative strain rate sensitivity (NSRS) and on the occurrence of serrated flow. We do this by comparing the ductility of “virtual” materials that have identical steady-state strain rate sensitivity but which differ in their tendency to exhibit serrated flow. Specifically, we propose a simple modification to the celebrated McCormick \cite{11} constitutive model of a dynamic strain aging material, which allows its transient response to be varied without affecting the steady-state behavior. Simple stability calculations, as well as numerical simulations, confirm that serrated flow can be suppressed if the transient response is sufficiently slow. The constitutive law is then used to model necking and PLC bands in a tensile bar deformed at constant nominal strain rate. We find that ductility (interpreted here as the critical strain required to initiate necking in a tensile test) in the NSRS regime is reduced relative to the ductility in regimes of positive strain rate sensitivity (PSRS). Furthermore, we find only a weak dependence of ductility on serrated flow that depends on specimen geometry. If the specimen is a near-perfect cylinder, a small decrease in the strain to localization is observed when deformation transitions from serrated flow to a stable mode. If a geometric defect is present in the specimen, however, suppressing serrated flow has no effect on the localization strain.

Therefore, we conclude that (i) to achieve high ductility, materials should be designed to minimize NSRS, without regard to PLC effects, and (ii) such material designs can be guided by models that need only to accurately predict NSRS, and do not need to capture the transient response of the material. Furthermore, our results suggest that strain localization in materials with negative strain rate sensitivity can be modeled using a constitutive model that has a sufficiently slow transient response to suppress PLC instabilities. This avoids the need to model short time-scale PLC bands in detail, and saves considerable computational expense.

The remainder of this paper is organized as follows. In the next section, we review the McCormick constitutive equations, and describe our modified treatment of its transient response. A stability analysis is used to calculate the conditions necessary for spatial and temporal instability in this material, confirming that a material with a slow transient response is stable. In Section 3, we then use fully three-dimensional (3-D) finite element simulations to model the deformation of a tensile specimen with circular cross-section. A small variation in cross-sectional area along the bar’s length is used to initiate necking in a controlled manner. We summarize our results in Section 4.

2. The McCormick constitutive equations with a delayed transient

McCormick’s constitutive equations are intended to model the behavior of a solute strengthened material which exhibits dynamic strain aging. The model assumes that plastic flow occurs as a result of thermally activated escape of dislocations that have been pinned by solute atoms. The plastic strain rate $\dot{\varepsilon}$ is related to stress $\sigma$ and the average local solute concentration $C$ near dislocations by

$$\dot{\varepsilon} = \dot{\varepsilon}_0 \exp \left( \frac{\sigma - \sigma_0(\varepsilon^p)}{S} - HC \right)$$

(1)

where $\dot{\varepsilon}_0$ is a characteristic strain rate, $S$ and $H$ are material constants controlling the instantaneous and steady-state strain rate sensitivity of the solid, while $\sigma_0(\varepsilon^p)$ is the zero temperature flow stress. The flow stress evolves with plastic strain due to strain hardening. Here, we adopt the following form for the strain hardening relation

$$\sigma_0(\varepsilon^p) = \sigma_{y0}(1 + \varepsilon^p / \varepsilon_0^p)^m$$

(2)

where $\{\sigma_{y0}, \varepsilon_0, m\}$ are material parameters, and $\varepsilon^p$ is the accumulated plastic strain. The solute concentration $C$ in Eq. (1) depends on the average age $t_d$ of dislocations according to

$$C = 1 - \exp \left( - \left( \frac{t_d}{t_d^*} \right)^{x} \right)$$

(3)

where $t_d$ is the characteristic time for solute diffusion across dislocations, $x$ is a phenomenological material constant and $t_d^*$ is the time that a representative mobile dislocation is pinned by obstacles. The age of dislocations $t_d$ evolves with time and strain rate according to the phenomenological kinetic law

$$\frac{dt_d}{dt} = 1 - \left( \frac{\dot{\varepsilon}^p t_d}{\Omega} \right)$$

(4)

It is important to note that the McCormick model is constructed based on an underlying physical phenomenon –
diffusion of solutes to a pinned dislocation – but that the details of the model are not derived from a mechanistic approach. Rather, negative strain rate sensitivity is built into the constitutive equations by construction. Furthermore, the kinetic law in Eq. (4) for the aging time is entirely ad hoc. More advanced constitutive laws derived from detailed analyses of physical mechanisms have been developed [4,5,9,10,12–15] and are preferable but more cumbersome to implement. The McCormick model, Eqs. (1)–(4), or the kinetic law Eq. (4) alone, has been used in a number of numerical studies [16–24] to examine macroscopic features and dynamics of a system exhibiting negative strain rate sensitivity. We use the model here for similar goals, i.e. to test the dependence of ductility on transverse strain rate sensitivity. We use the model here for microscopic features and dynamics of a system exhibiting negative strain rate sensitivity.

To be able to modify the transient kinetics of the strain rate response of the material, we generalize the McCormick model of Eq. (4) to a phenomenological kinetic law given by

\[ \frac{dt_a}{dt} = 1 - \left( \frac{\dot{\rho} \dot{t}_a}{\Omega} \right)^n \]

where \( n \) and \( \Omega \) are material constants. The latter constant quantifies the strain increment resulting from all the dislocations moving to the next obstacle along their path, and is related to the densities \( \rho_{\text{ms}}, \rho_f \) of mobile and forest dislocations and their Burgers vector \( b \) by \( \Omega = \rho_b b^2/\sqrt{\rho_f} \). With \( n = 1 \), Eq. (5) reduces to the standard McCormick constitutive law. During steady-state deformation at constant strain rate, the average dislocation age is \( \bar{t}_a^{(\infty)} = \Omega/\dot{\rho} \), so the steady-state flow behavior of the solid is independent of the new parameter \( n \).

In all the simulations reported here, we assume that the specimen is made from aluminum alloy AA5083 (a commercial Al-4.6% Mg alloy used extensively in automotive applications). Values of relevant material constants are listed in Table 1. Parameters \( \{\sigma_{\text{yy}}, t_0, m\} \), which describe strain hardening, were fitted to the measured flow curve for room-temperature deformation of AA5083 at the strain rate of \( 1.3 \times 10^3 \text{s}^{-1} \) reported by Benallal et al. [25]. The values of \( \{t_a, t_d, \alpha, S, H, \Omega\} \) , which govern the strain rate sensitivity of the solid, were selected to fit the flow stress vs. strain rate data reported by these authors. The flow stress during homogeneous deformation at constant strain rate is compared with their experimental data in Fig. 1a, while Fig. 1b shows the flow stress at 5% strain as a function of the strain rate. The stress–strain curve and the predicted regime of negative strain rate sensitivity are in good agreement with experiment.

The new parameter \( n \) in our constitutive law, which controls the transient aging behavior, cannot easily be fit to experiment. Its influence is shown in Fig. 1c and d, which shows the predicted evolution of flow stress following a step change in strain rate for several values of \( n \). Values of \( n < 1 \) delay the transient evolution of dislocation aging following a change in strain rate. As a result, the stress–strain curve approaches its new steady-state value more slowly. Note that the original McCormick model (with \( n = 1 \)) has a very rapid transient.

In addition to material parameters, an initial value must be assigned to the material state variable \( t_a \). In all the simulations reported here, we take \( t_a = \Omega/\dot{\rho} \) at the instant just before yield in the specimen, where \( \dot{\rho} \) is the applied nominal strain rate. This ensures that predicted stress–strain curves are not influenced by aging that occurs during the initial elastic loading of the specimen.

A simple estimate of the stability of the modified McCormick model can be obtained using the procedure developed by Mesarovic [26]. To this end, we consider a tensile specimen with Young’s modulus \( E \), cross-sectional area \( A \) and length \( l \), which is loaded by connecting the specimen to a spring of stiffness \( k \) and displacing the end of the spring at constant speed \( V \). Depending on the material properties and loading rate, we expect one of three types of behavior to occur: (i) the specimen may deform homogeneously at constant strain rate (stable behavior); (ii) the strain distribution in the specimen may develop spatial inhomogeneity (spatial instability); or (iii) the solid may deform homogeneously, but with a time varying strain rate (temporal instability). A perturbation calculation (described in detail by Mesarovic [26]) shows that the solid may develop spatial instabilities when

\[ \frac{d\sigma_0}{d\vartheta} - \sigma < -nS(1 - \Gamma)/\Omega \]

\[ \Gamma = Hx \left( \frac{t_a}{t_a} \right)^2 \exp \left\{ -x \left( \frac{t_a}{t_d} \right)^z \right\} \]

where \( t_a \) is the dislocation age in the specimen just prior to instability. If the specimen reaches a steady state during constant strain rate extension prior to instability, then the dislocation age is

\[ \bar{t}_a^{(\infty)} = \frac{\Omega}{V} \left[ 1 + \frac{1}{E} + \frac{A}{k} \left( \frac{d\sigma}{d\vartheta} - \frac{d\sigma_0}{d\vartheta} \right) \right] \approx \frac{\Omega}{V} \]

The condition for temporal instability reduces to

\[ \frac{EkI}{kl + AE} + \frac{d\sigma_0}{d\vartheta} - \sigma < -nS(1 - \Gamma)/\Omega \]
Except for the factor \( n \) on the right-hand side of the inequalities, conditions (6) and (8) are identical to the instability conditions given by Mesarovic [26]. It is helpful to note that the right hand sides of (6) and (8) can be expressed in terms of the steady-state strain rate sensitivity of the solid as

\[
\frac{nS(1 - \Gamma)}{\Omega} = - \frac{n}{\Omega} \frac{d\sigma_{ss}}{d\log(\dot{\varepsilon})}
\]

This shows that instabilities will occur most readily in materials with negative strain rate sensitivity. Furthermore, the effect of delaying the transient in the McCormick model by reducing the value of \( n \) is equivalent, from the perspective of stability, to reducing the negative steady-state strain rate sensitivity. Smaller \( n \) values therefore tend to stabilize the material, but without affecting its steady-state behavior.

### 3. Numerical simulations of PLC band formation and necking in a tensile specimen

We now address the role of serrated flow on localization during a tensile test. Our goal will be to compute the critical strain at the onset of necking during uniaxial straining of a tensile specimen at constant nominal strain rate. A full analysis of PLC instabilities and necking in a 3-D specimen can only be accomplished by means of numerical simula-

For this purpose, we have implemented a 3-D finite strain version of the modified McCormick constitutive equations (Eqs. (1)-(5)) in the explicit dynamic commercial finite element code ABAQUS/Explicit ver 6.7, through a user element. For completeness, the 3-D constitutive equations are summarized in Appendix A.

It is well known that the onset of necking in tensile specimens is sensitive to specimen geometry. To minimize the

![Finite element model of a uniaxial tensile specimen](image-url)

The specimen contains a small geometric defect, which reduces its cross-section near \( x_3 = L/2 \) (the defect is greatly exaggerated in the figure for clarity). The mesh contains 42,117 nodes.
number of parameters necessary to characterize the specimen, we consider a cylindrical tensile bar with length $L$ and circular cross-section, as shown in Fig. 2. Following Hutchinson and Neale [27], we introduce a small geometric imperfection in the specimen, which serves to initiate localization in a controlled manner. This geometric imperfection takes the form of a variation in the cross-sectional area of the bar along its length given by

$$A = A_0 \left[ 1 + \eta \cos \left( \frac{2\pi x_3}{L} \right) \right]$$

where the $A_0$ is the mean cross-sectional area of the undeformed bar and $\eta$ is the amplitude of the variation in cross-sectional area. The specimen is subjected to prescribed displacements representing a uniaxial tensile test at constant nominal strain rate. Specifically, we enforce $u_3 = 0$ on the plane at $x_3 = 0$ throughout straining, and constrain $u_1 = u_2$ at the origin. Axial rotation of the solid is constrained by enforcing $u_2 = 0$ at $x_3 = R, x_3 = 0$. The end of the specimen at $x_3 = L$ in the initial configuration is subjected to a constant velocity $V$ parallel to the $x_3$ direction.

A representative series of nominal stress vs. nominal strain curves are shown in Fig. 3. Results are shown for an applied nominal strain rate of $10^{-3}$ s$^{-1}$. Fig. 3a shows results for a specimen with a relatively large defect ($\eta = 0.05$), while Fig. 3b shows results for a nearly defect-free specimen ($\eta = 0.001$). In each case, a series of results are shown with different values of the exponent $n$ that controls the transient aging rate. Note that the curves have been offset vertically for clarity. The results show several interesting features. First, it is clear that small values of $n$, which delay the transient aging rate, suppress serrated flow, as expected. Secondly, the results show that, in a specimen with a large geometric defect, the stress–strain curves are essentially independent of $n$, and in particular there is no observable change in the strain to initiate necking when the deformation transitions between serrated flow and stable behavior. Results for a specimen with a small defect are more complex. Fig. 4 shows the ductility (which we take to be the critical strain required to initiate necking, identified by the strain corresponding to the peak stress in the nominal stress-nominal strain curve) as a function of $n$. Results are shown for a specimen deformed at nominal strain rate of $10^{-3}$ s$^{-1}$ (in the regime of NSRS) and for several values of the defect parameter $\eta$. The strain to localization decreases with increasing $\eta$, following the trend predicted by Hutchinson and Neale [27]. The localization strain is insensitive to $n$ (and hence to serrated flow) for $\eta \geq 0.05$. For small $\eta$, the localization strain actually drops slightly at the point where serrated flow is suppressed. This decrease may be preceded by an increase in localization strain. Nevertheless, the influence of $n$ on strain to localiza-
A closer examination of the deformation in the specimen during localization shows that PLC bands may act as precursors for necking, as suggested by Kang et al. [3] and Hopperstad et al. [4]. This behavior is only observed in specimens with a small geometric defect, however. For example, Fig. 5 shows the deformation and strain rate distribution in specimens with two different defect values. When the defect is small (Fig. 5a), localization occurs at random locations along the length of the specimen, initiated by geometric variations caused by PLC bands. In specimens with a larger defect (Figs. 5b), necking always initiates at the center of the specimen, where the cross-sectional area is a minimum.

In general, we conclude that serrated flow is not directly responsible for reducing ductility. Instead, reduced ductility is primarily associated with negative steady-state strain rate sensitivity, which is a necessary but not sufficient condition for serrated flow to occur. This is illustrated clearly in Fig. 6, which shows the strain to necking as a function of strain rate for a material with \( n = 0.01 \) (so no serrated flow occurs) and \( \eta = 0.01 \). A significant reduction in the ductility is observed in the regime of negative strain rate sensitivity. It is well known that increasing strain rate sensitivity tends to delay necking in materials with positive rate sensitivity [27]. Our results show that this behavior extends to materials in which the rate sensitivity is negative.

4. Conclusions

Over a characteristic range of strain rate and temperature, the flow stress of solute strengthened alloys decreases with increasing strain rate. Dynamic localization and serrated flow are often observed in materials with negative strain rate sensitivity, although NSRS is not by itself sufficient to cause serrated flow. Negative strain rate sensitivity is also accompanied by a significant loss of ductility in the material, which is manifested by a lower strain to initiate necking in a uniaxial tension test. The connection between serrated flow and this loss of ductility has been elusive.

We have used 3-D numerical simulations to examine the connection between serrated flow and necking in a uniaxial tension test, using a material constitutive law that allows for independent control of the steady-state rate sensitivity and the transient response upon a jump change in strain rate. We have shown, first, that a material with negative steady-state strain rate sensitivity need not necessarily develop PLC bands. This has allowed us to compare the behavior of two “virtual materials” which have identical negative steady-state rate sensitivity, but with one material deforming stably, and the other by nucleation and propagation of Portevin-Le Chatelier bands. As expected, a substantial decrease in the strain at the onset of necking was observed in the regime where the material has negative steady-state strain rate sensitivity. Importantly, suppressing serrated flow was found to have only a minor effect on ductility. We conclude that, although negative strain rate sensitivity
does reduce the ductility of a material during uniaxial tensile straining, this is not a direct consequence of serrated flow.

Of course, while PLC bands and serrated flow may not influence necking, this does not necessarily imply that they can be tolerated in a particular forming process. They may play an important role in nucleating damage in the material, for example; and may also affect the surface finish of a part. Serrated flow may also make a forming process difficult to control. These issues are beyond the scope of our computations, but the influence of transient material response on these issues is a promising area for further study.

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Appendix A

A.1. 3-D constitutive equations for use in finite element simulations

For numerical simulations, the constitutive equations outlined in the preceding section must be extended to general three-dimensional loading conditions, and to account for finite strains. We use a standard procedure to do this and provide a brief summary here for completeness.

Consider a representative material element within the solid. Let \( x \) denote the coordinates of the material particle (in a fixed, Cartesian basis) before deformation, and let \( u(x,t) \) denote the displacement of the material particle while the solid is loaded. Define the deformation gradient and its right polar decomposition by

\[
F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial x_j} \quad F_{ij} = R_{ik}U_{kj}
\]

where \( U_{ij} \) is symmetric and \( R_{ij} \) is a proper orthogonal tensor, which is taken to characterize the rotation of the material. We define the material spin as

\[
\Omega_{ij} = \bar{R}_{ik}R_{kj}
\]

The total deformation gradient is decomposed into elastic and plastic parts as

\[
F_{ij} = F_{ik}^eF_{kj}^p
\]

The velocity gradient \( \dot{L}_{ij} = \dot{F}_{ik}F_{kj}^{-1} \) can then be decomposed into elastic and plastic parts as

\[
\dot{L}_{ij} = \dot{F}_{ik}F_{kj}^{-1} = \dot{L}_{ij}^e + \dot{L}_{ij}^p \quad \dot{L}_{ij}^e = \dot{F}_{ik}^eF_{kj}^{-1}
\]

\[
\dot{L}_{ij}^p = F_{ik}^p\dot{F}_{kj}^{-1} = \dot{F}^{-1}_{mk}F_{mk}^{-1}
\]

The velocity gradient can be written as a sum of symmetric and skew symmetric parts, representing stretch rate and spin, as

\[
L_{ij} = D_{ij} + W_{ij} \quad D_{ij} = (L_{ij} + L_{ji})/2 \quad W_{ij} = (L_{ij} - L_{ji})/2
\]

\[
L_{ij} = D_{ij}^e + D_{ij}^p + W_{ij}^e + W_{ij}^p
\]

The stretch rate and spin can be decomposed into elastic and plastic parts as

\[
D_{ij}^e = (L_{ij}^e + L_{ji}^e)/2 \quad W_{ij}^e = (L_{ij}^e - L_{ji}^e)/2
\]

\[
D_{ij}^p = (L_{ij}^p + L_{ji}^p)/2 \quad W_{ij}^p = (L_{ij}^p - L_{ji}^p)/2
\]

It is convenient to characterize the internal stress in the solid using the Kirchhoff stress \( \tau_{ij} \), which is related to Cauchy (or “true”) stress \( \sigma_{ij} \) by \( \tau_{ij} = J\sigma_{ij} \), where \( J = \det (F_{ij}) \). The constitutive equations must then relate \( D_{ij}^e, D_{ij}^p, W_{ij}^e, W_{ij}^p \) to \( \tau_{ij} \) and its rate.

Here, we assume that both the elastic and plastic properties of the solid are isotropic. In this case, the elastic stretch rate is related to the time derivative of the Kirchhoff stress by

\[
\frac{d\tau_{ij}}{dt} + \tau_{ik}W_{kj}^e - W_{ij}^e\tau_{ij} = \frac{E}{1 + v}\left\{D_{ij}^e + \frac{v}{1-2v}D_{ik}^e\delta_{ij}\right\}
\]

where \( E \) and \( v \) denote the Young’s modulus and Poisson’s ratio of the solid, respectively. For convenience, we take the elastic spin to coincide with the material spin, so that \( W_{ij}^e = \Omega_{ij}, \quad W_{ij}^p = W_{ij} - \Omega_{ij} \). Under these conditions, Eq. (17) can be regarded as relating the rate of change of stress components \( \dot{\tau}_{ij}^e = R_{ik}\tau_{kj} \) in a basis that rotates with the material spin to the components of stretch rate \( \dot{D}_{ij}^e = R_{ik}\dot{D}_{kj}^e \) in the same basis as follows

\[
\frac{d\tau_{ij}^e}{dt} = \frac{E}{1 + v}\left\{\dot{D}_{ij}^e + \frac{v}{1-2v}\dot{D}_{ik}^e\delta_{ij}\right\}
\]

This form of the elastic constitutive equation is particularly convenient for numerical simulations. Finally, the plastic stretch rate is related to stress by

\[
\dot{D}_{ij}^p = \dot{\varepsilon}(\tau_{ij}, t) \frac{3\tau_{ij}}{2\tau_e} \quad \dot{\varepsilon}_{ij} = \tau_{ij} - \tau_{kk}\delta_{ij}/3 \quad \tau_e = \sqrt{3\tau_{ij}\tau_{ij}^e}/2
\]

where \( \varepsilon(\sigma, t) \) is the relationship between stress and strain rate detailed in Section 2.

References