Implemention of a beam element in finite element analysis

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1. Topic:

A beam under point loads is solved. Two-node beam element is implemented.

2. Theory[1]:

The basic constitutive equation is:

$$\int_0^L EI\frac{d^4w}{dx^4} + q = 0$$

The boundary condition is:

$$m^* = -EI\frac{d^2w}{dx^2}$$

$$V^* = -EI\frac{d^3w}{dx^3}$$

where, E is the Young's modulus of the beam, I is the moment of area, L is the length of the beam, w is the deflection of the beam, q is the load, m* is the momentum, and V* is the shear force.

So the weak form is:

$$\int_0^L EI\frac{d^4w}{dx^4}\delta w dx + \int_0^L f\delta w\, dx - \left[\left(V^* + EI\frac{d^3w}{dx^3}\right)\delta w\right]_0^L - \left[\frac{d\delta w}{dx}\left(m^* - EI\frac{d^2w}{dx^2}\right)\right] = 0$$

after integration by part of the basic constitutive equation to get the principle of virtual work:

$$\int_0^L EI\frac{d^2w}{dx^2}\frac{d^2\delta w}{dx^2}dx + \int_0^L f\delta w\, dx - [V^*\delta w]_0^L - \left[\frac{d\delta w}{dx}m^*\right] = 0$$

Assume

$$w = a_1x^3 + a_2x^2 + a_3x + a_4$$

So,

$$w(0) = w_1 = a_4$$

$$\frac{dw(0)}{dx} = \Phi_1 = a_3$$

$$w(Le) = w_2 = a_1Le^3 + a_2Le^2 + a_3Le$$

$$\frac{dw(Le)}{dx} = \Phi_2 = 3a_1Le^2 + 2a_2Le$$

Where w_1 is the deflection of node 1, w_2 is the deflection of node 2, Φ_1 and Φ_2 are the rotation angle of node 1 and 2.

Now,

$$w = \left[\frac{2}{Le^3}(w_1 - w_2) + \frac{1}{Le^2}(\Phi_1 - \Phi_2)\right]x^3 + \left[-\frac{3}{Le^2}(w_1 - w_2) - \frac{1}{Le}(2\Phi_1 - \Phi_2)\right]x^2 + \Phi_1\mathrm{x} + w_1$$

$$w = [N]\{d\}$$

Where

$$N_1 = \frac{1}{Le^3}(2x^2 - 3x^2Le + Le^3)$$
$$N_2 = \frac{1}{Le^3}(x^3Le - 2x^2Le^2 + xLe^3)$$
$$N_3 = \frac{1}{Le^3}(-2x^3 + 3x^2Le)$$
$$N_4 = \frac{1}{Le^3}(x^3Le - x^2Le^2)$$
$$\{d\} = \begin{Bmatrix} w_1 \\ \Phi_1 \\ w_2 \\ \Phi_2 \end{Bmatrix}$$

These cubic shape (or interpolation) functions are known as Hermite cubic interpolation functions. Using the shape function and the weak form of the beam equation, the elemental stiffness is

$$\text{ke}=\frac{EI}{Le^3}\begin{pmatrix} 12 & 6Le & -12 & 6Le \\ 6Le & 4Le^2 & -6Le & 2Le^2 \\ -12 & -6Le & 12 & -6Le \\ 6Le & 2Le^2 & -6Le & 4Le^2 \end{pmatrix}$$

The element equation is

$$ke = \{d\}fe$$

The residual forces vector is

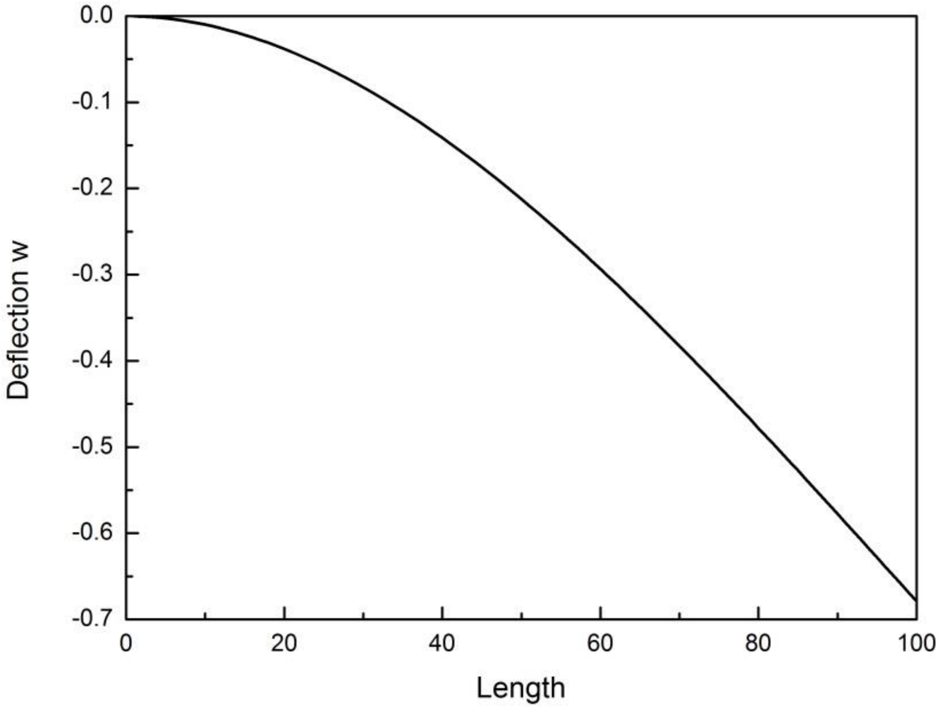
$$fe = \int_0^{Le} -[N]^T q dx + ([N]^T,_x m^* - [N]^T V^*)_0^{Le}$$

3. Analysis:

Now test the above code with a simple problem of a beam with one end fixed at x=0: consider a beam with a circular cross-section of 10 diameter and a length of 100. The Young’s Modulus of the beam is 10^5. There is a load of 100 acting in the -y direction at the right end of the beam. The maximum deflection of the beam is -0.6791 at L=100. While for the analytical solution,

$$\text{I}=\frac{\pi\left(\frac{d}{2}\right)^4}{4} \qquad \text{W}=\frac{FL^3}{3EI}$$

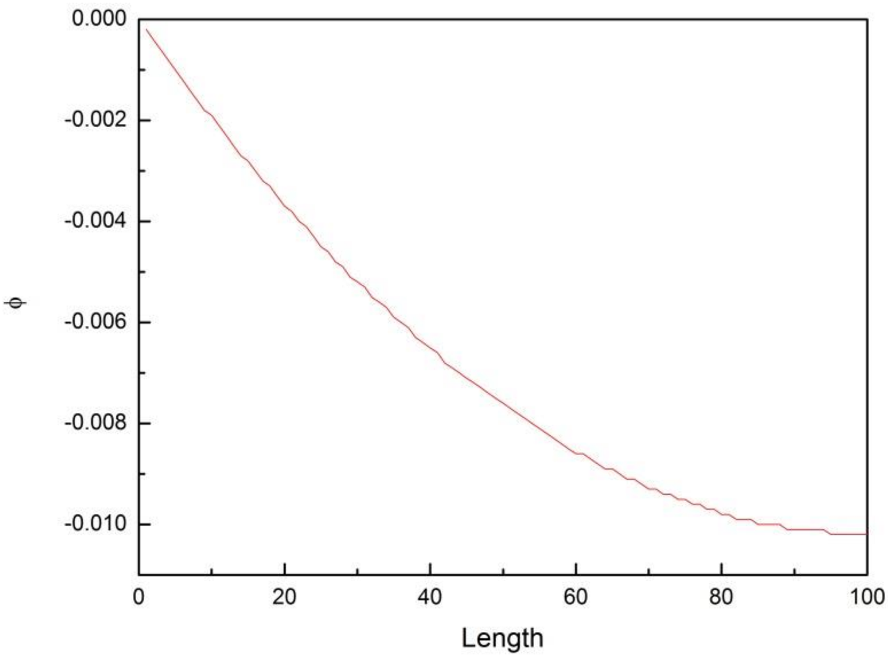
So the maximum deflection is -0.67906, which agrees with the FEA solution.



The rotation angle of the beam is -0.0102 at L=100. While for the analytical solution,

$$\Phi = \frac{FL^2}{2EI}$$

So the rotation angle of the beam is -0.01019, which is pretty close to the FEA result.



4. Conclusion:

Finite element analysis is implemented to approximate the beam deflection. Cubic shape functions are used. The numerical results agree with the analytical results.

The beam theory solution predicts a quartic (fourth-order) polynomial expression for a beam subjected to uniformly distributed loading, while the FEA solution assumes a cubic (third-order) displacement behavior in each beam all load conditions. So the FE solution predicts a stiffer structure than the actual one. However, as the number of nodes increased, better solution could be got.

Reference:

1. Daryl L. Logan , *A First Course in the Finite Element Method*.