ENGN2340 Final Project

Reduced Integration with Hourglass Control

Daren Liu

Introduction:

In class we discussed the topic of volumetric locking problem. Consider a long hollow cylinder with an internal pressure and thus deforming in plane strain. When Poisson's ratio ν goes toward 0.5, the problem tends to an incompressible limit, and the finite element solution highly underestimates the exact solution.

The basic reason for volumetric locking is that incompressibility imposes a constraint on deformation mode, and the easiest way to avoid the locking is to reduce the number of integration points. However, "Reduced Integration Method" fails for four-node quadrilateral elements and eight-node brick elements. It can be easily demonstrated that hourglassing would appear, due to the fact that the stiffness matrix is nearly singular, and the system of equations includes a weakly constrained deformation mode.

Of course we can use "Selectively Reduced Integration" or "B-Bar Method" to solve hourglassing, but this project focuses on "Hourglass Control" in reduced integration. It is done by adding an artificial stiffness to the element in order to constrain the hourglass mode.

Procedure:

1. Define the "hourglass base vectors" $\Gamma^{a(i)}$, which specifies the displacements of the ath node in the ith hourglass mode.
   For four-noded quadrilateral elements:

$$\Gamma^{a(1)} = (+1,-1,+1,-1)$$

   For eight-noded brick elements:

$$\Gamma^{a(1)} = (+1,+1,-1,-1,-1,-1,+1,+1)$$

$$\Gamma^{a(2)} = (+1,-1,-1,+1,-1,+1,+1,-1)$$

$$\Gamma^{a(3)} = (+1,-1,+1,-1,+1,-1,+1,-1)$$

$$\Gamma^{a(4)} = (-1,+1,-1,+1,+1,-1,+1,-1)$$

2. For each mode, calculate the "hourglass shape vectors" $\gamma^{a(i)}$ as follows:

$$\gamma^{a(i)} = \Gamma^{a(i)} - \frac{\partial N^a(\xi = 0)}{\partial x_j} \sum_{b=1}^{N_e} \Gamma^{b(i)} x_j^b$$

Where N_e is the number of nodes on the element and $\frac{\partial N^a(\xi = 0)}{\partial x_j}$ is the value of $\frac{\partial N^a}{\partial x_j}$ at $\xi = 0$.

3. Modify the expression for element stiffness matrix:

$$k_{aibk}^{(l)} = \int_{V_e^{(l)}} C_{ijkl} \frac{\partial N^a(x)}{\partial x_j} \frac{\partial N^b(x)}{\partial x_l} dV + \kappa V_e^{(l)} \sum_m \gamma^{a(m)} \gamma^{b(m)}$$

Here V_e is element volume, V_e = dtm*w(i).

Numerical parameter κ controls the stiffness of the hourglass resistance. We can take $\kappa = 0.01\mu \frac{\partial N^a}{\partial x_j} \frac{\partial N^a}{\partial x_j}$, with μ being the elastic shear modulus.

FEACHEAP Implementation:

In the following, I implement the above scheme in FEACHEAP with linear elastic four-node 2D quadrilateral element and eight-node 3D brick element.

Consider the same pressurized cylinder problem as in the B-bar homework. Set the inner pressure as 1 and use the same meshes. As for the material parameters, take Shear Modulus to be 1 and Poisson's Ratio to be 0.499, which is a close-incompressible scenario. The initial mesh is plotted as below.

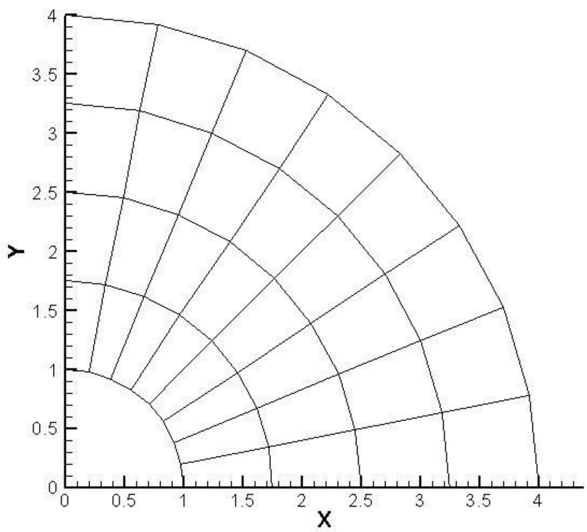

Figure 1. Initial mesh

Now calculate the deformation using Reduced Integration Method. The results look like complete garbage.

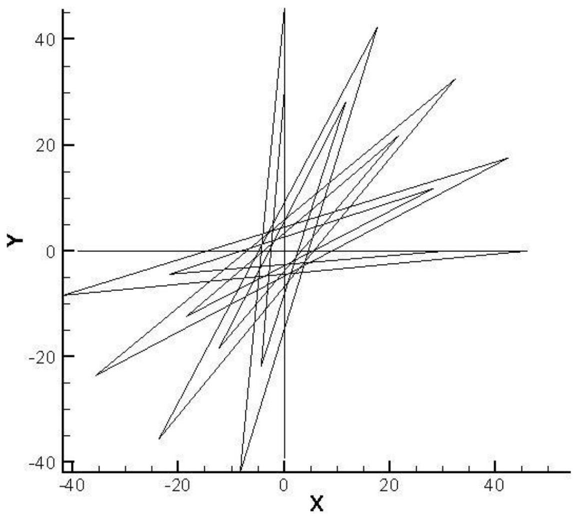

Figure 2. Solution of displacements using Reduced Integration Method

Scale the pressure down to one percent of the original value and plot the deformed meshes again, we can see hourglassing showing up here. This proves that Reduced Integration does not work for four-node quadrilateral elements.

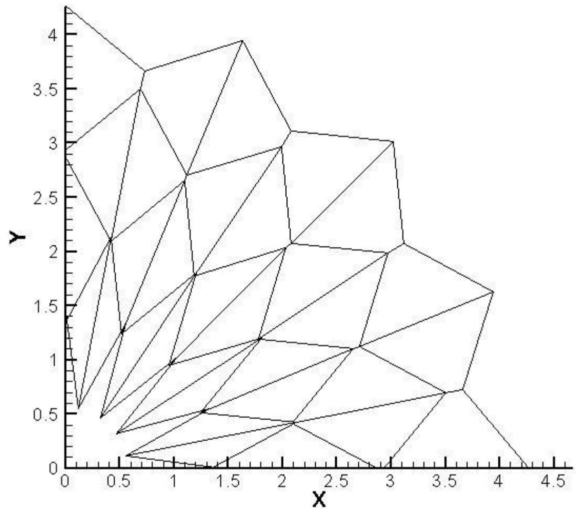

Figure 3. Hourglassing under scaled force using Reduced Integration Method

Now implement the above Hourglass Control theory and plot again. Comparing the plots with those of B-bar method, we see almost exactly the same results.

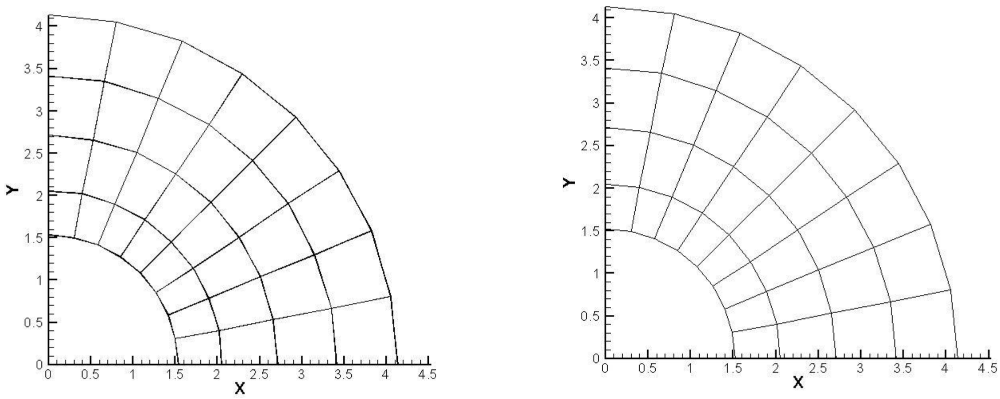

Figure 4. Comparison of deformed meshes
(Hourglass Control on the left, B-bar method on the right. Same in the following)

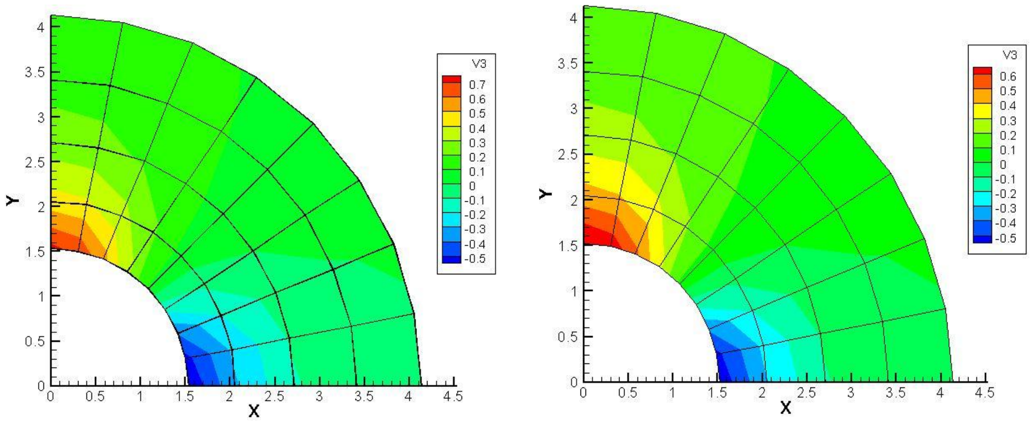

Figure 5. Comparison of σ_{11}

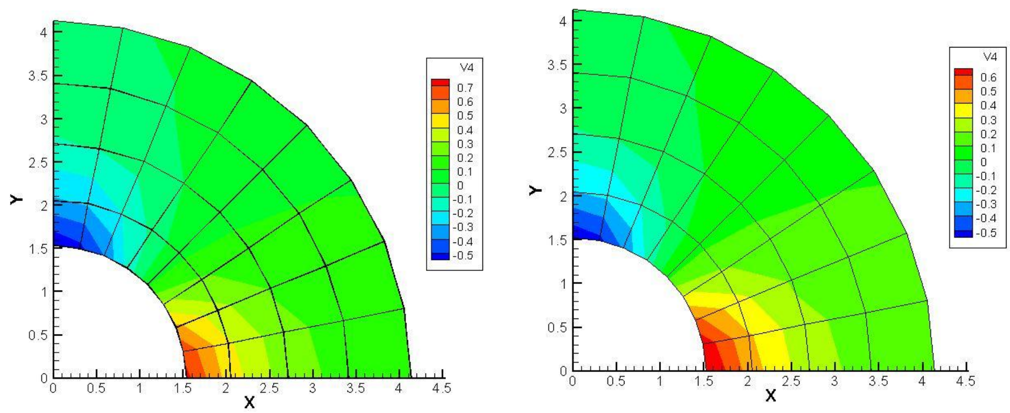

Figure 6. Comparison of σ_{22}

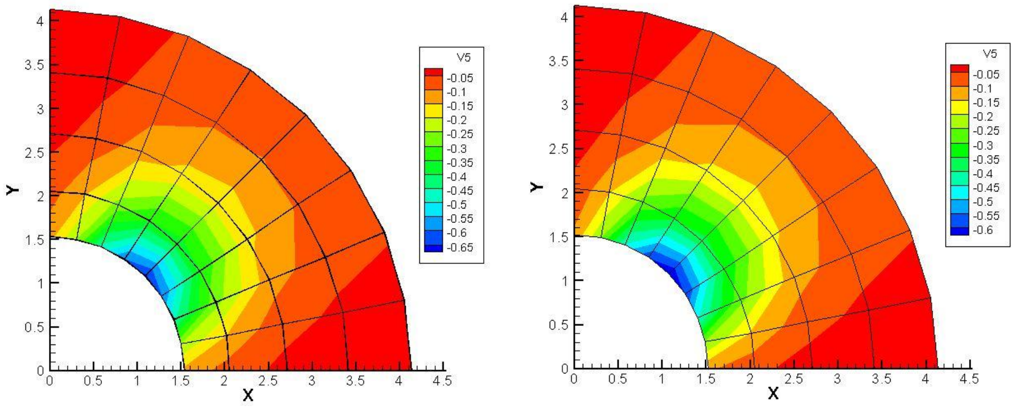

Figure 7. Comparison of σ_{12}

Obviously hourglassing can be cured here, so “Hourglass Control” indeed eliminates hourglassing in four-node quadrilateral element.

Same phenomena can be observed in 3D brick element. Below is the comparison before and after hourglass control

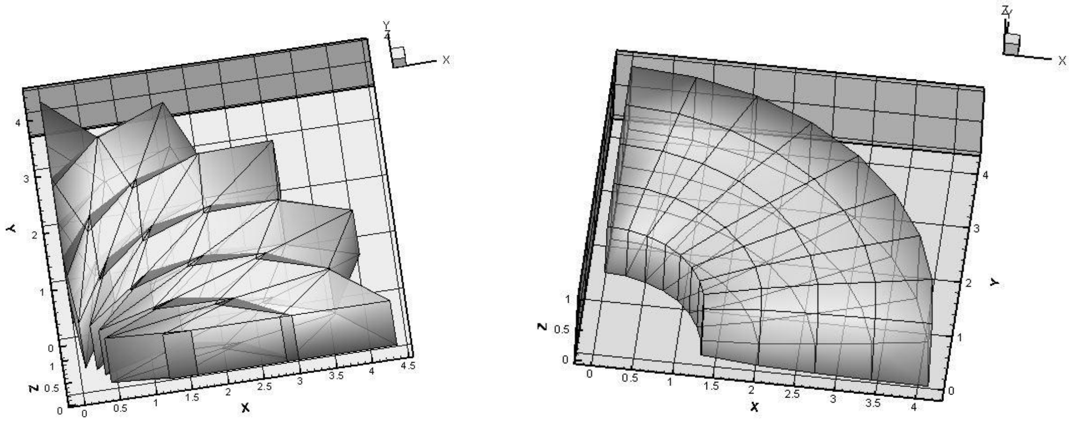

Figure 8. Scaled hourglassing (left) and controlled results (right) for eight-node 3D brick element

Reference:

Applied Mechanics of Solids, by Allan F. Bower.