Finite Strain Elastic-Viscoplastic Model

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1 Introduction

The main goal of the project is to model finite strain rate-dependent plasticity using a model compatible for high strain rates. In such scenarios, Johnson-Cook model is very useful. The model is used in adiabatic dynamic simulations, for example, pressure-shear plate impact experiments and machining. The model incorporates temperature effects as well, using a power-law dependence. Temperature dependence is ignored in the present study.

In the present study, the model is developed keeping in mind the future use for pressure-shear impact simulations. Kinematics of pressure-shear experiment is introduced to give an idea about the type of deformation involved. This is followed by introduction to the model and simulations on two elements. The FEA formulation is done in EN234FEA.

2 Governing Equations

2.1 Kinematics

Deformation in a pressure-shear experiment can be written as:

$$x_1 = \lambda(t)X_1 \tag{1}$$
$$x_2 = X_2 - \kappa(t)X_1 \tag{2}$$
$$x_3 = X_3 \tag{3}$$

The deformation gradient and velocity gradients are, therefore:

$$\mathbf{F} = \begin{bmatrix} \lambda(t) & 0 & 0 \\ -\kappa(t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \begin{bmatrix} \dot{\lambda}\lambda & 0 & 0 \\ -\dot{\kappa}/\lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5}$$

2.2 Finite Strain Viscoplastic Material Model

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{6}$$
$$\mathbf{L} = \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1} + \mathbf{F}^e\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}(\mathbf{F}^e)^{-1} \tag{7}$$
$$= \mathbf{L}^e + \mathbf{L}^p = (\mathbf{D}^e + \mathbf{W}^e) + (\mathbf{D}^p + \mathbf{W}^p) \tag{8}$$

Here, $\mathbf{W}^p = 0$ is considered. Consider the Kirchoff stress ($\boldsymbol{\tau} = J * \boldsymbol{\sigma}$) as the stress-measure for this study. Rate of change of Kirchhoff stress is defined as:

$$\dot{\boldsymbol{\tau}} = \hat{\boldsymbol{\tau}}^e + (W^e\boldsymbol{\tau} - \boldsymbol{\tau}W^e) \tag{9}$$

where $\hat{\boldsymbol{\tau}}$ is Jaumann stress rate and is given as:

$$\hat{\boldsymbol{\tau}} = \mathbf{C}^e : \mathbf{D}^e + (\mathbf{D}^e \boldsymbol{\tau} + \boldsymbol{\tau} \mathbf{D}^e) \tag{10}$$

The second term of the Kirchoff stress rate is usually taken care of by ABAQUS, so we need to find the Jaumann stress rate only.

Plasticity Equations

$$\mathbf{D}^p = \frac{3}{2} \frac{\dot{\epsilon}_e}{\tau_e} \boldsymbol{\tau}^D \tag{11}$$

$$\tau_e = \sqrt{\frac{3}{2} \boldsymbol{\tau}_{ij}^D \boldsymbol{\tau}_{ij}^D} \tag{12}$$

$$\dot{\epsilon}_e = \sqrt{\frac{2}{3} \mathbf{D}_{ij}^p \mathbf{D}_{ij}^p} \tag{13}$$

A constitutive law governing $\dot{\epsilon}_e$ is required. One of the laws particularly useful for impact problems is the Johnson/Cook model, where the yield stress is given as:

$$\sigma_y = (A + B\epsilon_e^n) \left[1 + C\, ln \left(\frac{\dot{\epsilon}_e}{\dot{\epsilon}_0} \right) \right] \left[1 - \left(\frac{T - T_0}{T_m - T_0} \right)^m \right] \tag{14}$$

where $\sigma_e = \sqrt{\frac{3}{2} \mathbf{s}_{ij} \mathbf{s}_{ij}} = \frac{1}{J} \sqrt{\frac{3}{2} \boldsymbol{\tau}_{ij}^D \boldsymbol{\tau}_{ij}^D}$. A is the static shear strength, B is the strain-hardening modulus, $C$is the rate-sensitivity coefficient, m is the thermal-softening exponent, n is the strain-hardening exponent, T is the current temperature, T_0 is the room temperature and T_m is the melting temperature. Ignoring the effects of temperature, i.e. assuming $T = T_0$,

$$\dot{\epsilon}_e = \dot{\epsilon}_0 e^{\left[\frac{1}{C} \left(\frac{\sigma_e}{A + B\epsilon_e^n} - 1 \right) \right]} \tag{15}$$

Since the strain rate grows exponentially with effective shear stress, it is necessary to limit the strain rate to deal with high stresses during initial elastic response. A limiting strain rate $\dot{\epsilon}_e^{lim}$ is used as follows to define the actual plastic strain rate:

$$\dot{\epsilon}_e^{eff} = \frac{\dot{\epsilon}_e^{lim} \dot{\epsilon}_e}{\dot{\epsilon}_e + \dot{\epsilon}_e^{lim}} \tag{16}$$

Johnson-Cook Dynamic Failure Criterion

A damage parameter is calculated at the integration points and failure is assumed to occur when this parameter is equal to 1. The damage parameter,ω is given as:

$$\omega = \sum \frac{\Delta \epsilon_e}{\epsilon_{e,f}} \tag{17}$$

where $\epsilon_{e,f}$ is the failure plastic strain. The summation is performed over all the time increments in the analysis. The failure plastic strain is assumed to be dependent on the plastic strain rate in similar fashion as the yield stress and is formulated as below:

$$\epsilon_{e,f} = (d_1 + d_2\, e^{(d_3 \frac{p}{\sigma_e})}) \left[1 + d_4\, ln \left(\frac{\dot{\epsilon}_e}{\dot{\epsilon}_0} \right) \right] \left[1 + d_5 \left(\frac{T - T_0}{T_m - T_0} \right) \right] \tag{18}$$

The parameters d_1 to d_5 are failure parameters determined using experiments.

Simplification of the Jaumann stress rate:

$$\hat{\boldsymbol{\tau}^e_{ij}} = \left[\frac{E}{1+\nu}\mathbf{D}_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)}\mathbf{D}_{kk}\delta_{ij}\right] - \left[\frac{3}{2}\frac{E}{(1+\nu)}\frac{\dot{\epsilon}^{eff}_e}{\tau_e}\boldsymbol{\tau}^D_{ij}\right] + \mathbf{M}_{ij} \tag{19}$$

$$\mathbf{M}_{ij} = \mathbf{D}_{ik}\boldsymbol{\tau}_{kj} + \boldsymbol{\tau}_{im}\mathbf{D}_{mj} - \frac{3}{2}\frac{\dot{\epsilon}^{eff}_e}{\tau_e}(\boldsymbol{\tau}^D_{ik}\boldsymbol{\tau}_{kj} + \boldsymbol{\tau}_{im}\boldsymbol{\tau}^D_{mj}) \tag{20}$$

Hence, the Kirchoff stress rate, calculated explicitly is given as:

$$\begin{aligned}\boldsymbol{\tau}^{(n+1)}_{ij} = \boldsymbol{\tau}^{(n)}_{ij} &+ \left[\frac{E}{1+\nu}\int_{t_n}^{t_{n+1}}\mathbf{D}_{ij}dt + \frac{E\nu}{(1+\nu)(1-2\nu)}\int_{t_n}^{t_{n+1}}\mathbf{D}^{(n)}_{kk}\delta_{ij}dt\right] \\ &- \left[\frac{3}{2}\frac{E}{(1+\nu)}\frac{\Delta t\ \dot{\epsilon}^{eff(n)}_e}{\tau_e}\boldsymbol{\tau}^{D(n)}_{ij}\right] + \int_{t_n}^{t_{n+1}}\mathbf{M}^{(n)}_{ij}dt + \int_{t_n}^{t_{n+1}}\mathbf{Q}^{(n)}_{ij}dt\end{aligned} \tag{21}$$

where

$$\mathbf{Q}_{ij} = \mathbf{W}_{ik}\boldsymbol{\tau}_{kj} - \boldsymbol{\tau}_{im}\mathbf{W}_{mj} \tag{22}$$

Derivation

$$\hat{\boldsymbol{\tau}^e} = \mathbf{C}^e : \mathbf{D}^e + \mathbf{M} \tag{23}$$

$$\mathbf{C}^e : \mathbf{D}^e = \mathbf{C}^e : \mathbf{D} - \mathbf{C}^e : \mathbf{D}^p \tag{24}$$

$$(\mathbf{C}^e : \mathbf{D})_{ij} = \mathbf{C}^e_{ijkl}\mathbf{D}_{kl} = \left[\frac{E}{2(1+\nu)}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{E\nu}{(1+\nu)(1-2\nu)}\delta_{ij}\delta_{kl}\right]\mathbf{D}_{kl} \tag{25}$$

$$= \left[\frac{E}{(1+\nu)}\mathbf{D}_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)}\mathbf{D}_{kk}\delta_{ij}\right] \tag{26}$$

$$(\mathbf{C}^e : \mathbf{D}^p)_{ij} = \frac{E}{2(1+\nu)}\frac{3\dot{\epsilon}_e}{2\tau_e}(\boldsymbol{\tau}^D_{ij} + \boldsymbol{\tau}^D_{ji}) = \frac{E}{(1+\nu)}\frac{3\dot{\epsilon}_e}{2\tau_e}(\boldsymbol{\tau}^D_{ij}) \tag{27}$$

$$\mathbf{M}_{ij} = \mathbf{D}^e_{ik}\boldsymbol{\tau}_{kj} + \boldsymbol{\tau}_{ik}\mathbf{D}^e_{kj} \tag{28}$$

3 Newton-Raphson for $\Delta\epsilon_e$

Here, $\Delta\epsilon^{eff}_e$ is addressed simply as $\Delta\epsilon_e$ and $\boldsymbol{\epsilon}^p_{ij}$ and $\mathbf{D}^p_{ij}$ are the same thing.

$$\dot{\boldsymbol{\epsilon}}_{ij} = \dot{\boldsymbol{\epsilon}}^e_{ij} + \dot{\boldsymbol{\epsilon}}^p_{ij} \tag{29}$$

This leads to

$$\dot{\mathbf{s}}_{ij} = \frac{E}{1+\nu}\dot{\mathbf{e}}^e_{ij} \tag{30}$$

$$\mathbf{s}^{(n+1)}_{ij} = \mathbf{s}^{(n)}_{ij} + \frac{E}{1+\nu}(\Delta\mathbf{e}_{ij} - \Delta\boldsymbol{\epsilon}^p_{ij}) \tag{31}$$

Elastic Predictor

$$\mathbf{s}^{*(n+1)}_{ij} = \mathbf{s}^{(n)}_{ij} + \frac{E}{1+\nu}\Delta\mathbf{e}^{(n)}_{ij} \tag{32}$$

$$\sigma^{*(n+1)}_e = \sqrt{\frac{3}{2}\mathbf{s}^{*(n+1)}_{ij}\mathbf{s}^{*(n+1)}_{ij}} \tag{33}$$

Correction: Let $\mathbf{s}_{ij}^{(n+1)} = \beta \mathbf{s}_{ij}^{*(n+1)}$. On solving, β can be found to be

$$\beta = 1 - \frac{3E}{2(1+\nu)} \frac{\Delta\epsilon_e}{\sigma_e^{*(n+1)}} \tag{34}$$

Now we try to solve for $\Delta\epsilon_e$.

$$\Delta\epsilon_e = \frac{\Delta t \dot{\epsilon}^{lim} \dot{\epsilon}_0 e^{\frac{1}{C}\left(\frac{\sigma_e^{(n+1)}}{A+B\epsilon_e^{(n+1)n}}-1\right)}}{\dot{\epsilon}^{lim} + \dot{\epsilon}_0 e^{\frac{1}{C}\left(\frac{\sigma_e^{(n+1)}}{A+B\epsilon_e^{(n+1)n}}-1\right)}} = \frac{\dot{\epsilon}^{lim}\Delta t}{1 + \frac{\dot{\epsilon}^{lim}}{\dot{\epsilon}_0} e^{\frac{1}{C}\left(1-\frac{\sigma_e^{(n+1)}}{A+B\epsilon_e^{(n+1)n}}\right)}} \tag{35}$$

$$F = 1 - \frac{\dot{\epsilon}^{lim}\Delta t}{\Delta\epsilon_e} + \frac{\dot{\epsilon}^{lim}}{\dot{\epsilon}_0} e^{\frac{1}{C}\left(1-\frac{\sigma_e^{(n+1)}}{A+B\epsilon_e^{(n+1)n}}\right)} \tag{36}$$

$$\frac{dF}{d\Delta\epsilon_e} = \frac{\dot{\epsilon}^{lim}\Delta t}{\Delta\epsilon_e^2} + \frac{\dot{\epsilon}^{lim}}{\dot{\epsilon}_0} e^{\frac{1}{C}\left(1-\frac{\beta\sigma_e^{*(n+1)}}{A+B(\epsilon_e^{(n)}+\Delta\epsilon_e)^n}\right)} \left[\frac{1}{C}\left(\frac{3E}{2(1+\nu)}\frac{1}{A+B(\epsilon_e^{(n)}+\Delta\epsilon_e)^n} + \frac{\beta\sigma_e^{*(n+1)} Bn * (\epsilon_e^{(n)}+\Delta\epsilon_e)^{n-1}}{(A+B(\epsilon_e^{(n)}+\Delta\epsilon_e)^n)^2}\right)\right] \tag{37}$$

F can also be formulated as:

$$F = \sigma_e^{(n+1)} - (A + B\,\epsilon_e^{(n+1)n})\left[1 + C\,ln\left(\frac{\dot{\epsilon}_e}{\dot{\epsilon}_0}\right)\right] \tag{38}$$

4 FE Formulation

Finite Element formulation used is similar to Gurson model implemented in Assignment-10. The UEL is written for finite strain using L-bar method.

Maximum time step that can be used is given by:

$$\Delta t_{max} = \frac{L^e}{c} \tag{39}$$

where L^e is length of the element and c is longitudinal wave speed in the material. The material parameters used in the simulations correspond to Al-6061-T6: E =70 GPa, A =324.1 MPa, B =113.8 MPa, C =0.002 MPa, n =0.42, $\dot{\epsilon}_0$ =1,$\dot{\epsilon}^{lim}$ =1, d_1 =-0.77, d_2 =1.45, d_3 =0.47, d_4 =0. Similarly, for Steel 4340, the following parameters are available: E =200 GPa, A =792 MPa, B =510 MPa, C =0.014 MPa, n =0.26, $\dot{\epsilon}_0$ =1,$\dot{\epsilon}^{lim}$ =1, d_1 =0.05, d_2 =3.44, d_3 =2.12, d_4 =0.002.

A new subroutine, *stress_update_pressureshear* is written and the plastic strain increment is solved implicitly using Newton-Raphson. It can be solved by either of the two formulations of F presented above. Damage variable, ω is stored as the ninth of the 10 state variables (the first six being Kirchhoff stresses followed by the old accumulated plastic strain, new accumulated plastic strain and e11). **NOTE:**Element deletion is included in the subroutine *el_pressureshear*. The values at integration points are projected onto the nodes using *fieldvars_pressureshear*. The corresponding input files are *PressureShear_3D.in* and *notch_fracture_dynamic.in*

5 Results

The model is tested on two-elements. A unit displacement is applied in x-direction at $t = 0$. Stress and strain in x-direction,damage and accumulated plastic strain are plotted in Figure 1. The time step chosen is $1.d - 5$.

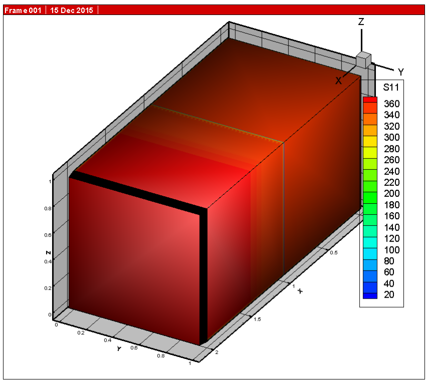
Newton-Raphson checks are performed using a 1D MATLAB code prior to these simulations. The model is also applied to Mode-I notch fracture problem for Gurson Model as shown in Figure 2.

6 References

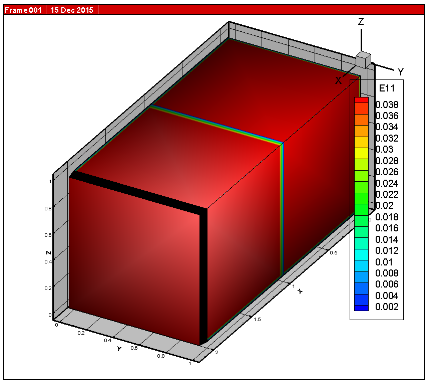
1. Stephen.E.Grunschel, *Pressure-Shear Plate Impact Experiments on High-Purity Aluminum at Temperatures Approaching Melt*, PhD Thesis, 2009.

2. ABAQUS Documentation, 18.2.7, *Johnson Cook plasticity.*

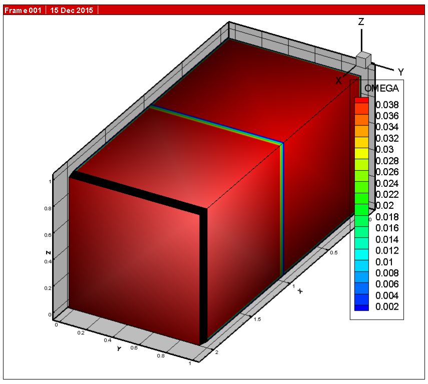
3. ABAQUS Technology Brief, Dassault Systems. *Simulation of the ballistic perforation of aluminum plates with Abaqus/Explicit.*



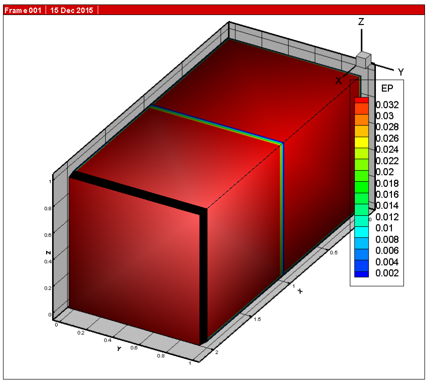
(a) s_{11}



(b) e_{11}

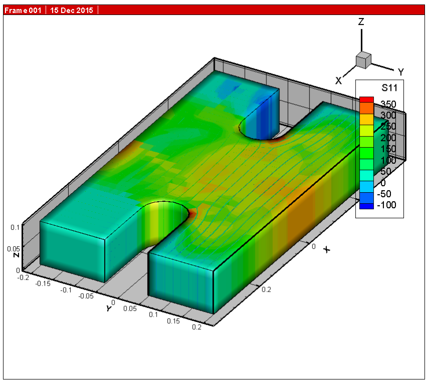


(c) Damage parameter, ω

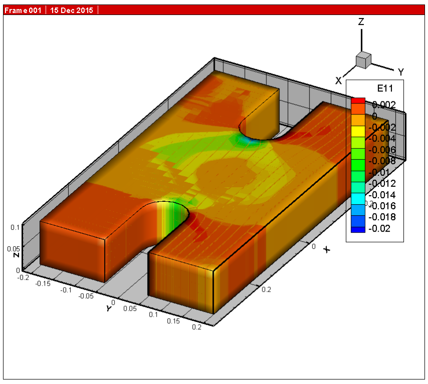


(d) Accumulated Plastic Strain

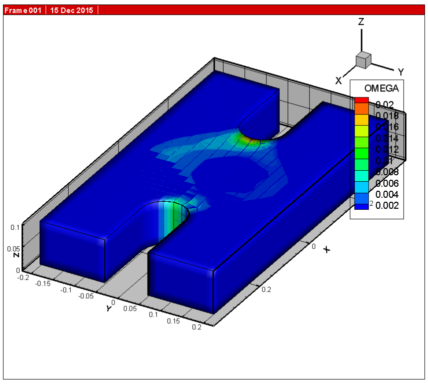
Figure 1: 80,000 time steps. Each time step equal to $1.d-5$



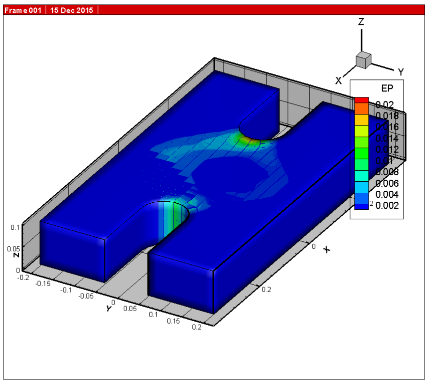
(a) s_{11}



(b) e_{11}



(c) Damage parameter, ω



(d) Accumulated Plastic Strain

Figure 2: 3000 time steps. Each time step equal to $1.d-5$