Implement of Augmented Lagrangian Hybrid element

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Background and Constitutive Equation

Augmented Lagrangian Hybrid element is a way to fix volume locking caused by approaching incompressible limit. The basic idea is calculating the hydrostatic pressure and deviatoric stress separately and adding fake nodes to each element to be pressure nodes.

$$\sigma_{ij} = S_{ij} + P\delta_{ij} \quad P = \delta_{kk}/3$$

$$S_{ij} = C^{DEV}_{ijkl} * \varepsilon_{kl}$$
$$P = K * \varepsilon_{kk}$$
$$K = E/3*(1-2\upsilon)$$

Constitutive Equation

$$\partial\sigma_{ij}/\partial x_j + b_i = 0$$
$$\sigma_{ij} n_j = t_i^* on S_2$$
$$K*\varepsilon_{kk} - P = 0$$

Weak form and using interpolation:

$$\int_V C^{DEV}_{ijkl} \partial\eta_i/\partial x_j * \partial u_k/\partial x_l dV + \int_V P \partial\eta_i/\partial x_j dV - \int_V b_i \eta_i dV - \int_{S_2} t^*{}_i \eta_i dA = 0$$

$$\int_V (K\partial u_k/\partial x_k - P) * \mathrm{q}/\mathrm{K}\, dV = 0$$

$$\forall q \forall admiss \eta_i$$
$$u_i = N^a u_i^a$$
$$P = m^a P^a$$
$$\eta_i = N^a \eta_i^a$$
$$q = m^a q^a$$

M here should be one order less than N

We can get element Residual Force:

$$r^{el} = \int_{vel} B^T \sigma \mathrm{dV}$$

Element stiffness：

$$\begin{pmatrix} K^{uu} & K^{up} \\ K^{pu} & K^{pp} \end{pmatrix} \begin{pmatrix} u \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix}$$

$$K^{\mathrm{uu}} = \int_{V\mathrm{el}} B^T D^{DEV} B \mathrm{d}V$$

$$K^{\mathrm{up}} = \int_{V\mathrm{el}} B^T [\mathrm{m}] \mathrm{d}V$$

$$K^{\mathrm{pu}} = \int_{V\mathrm{el}} [\mathrm{m}]^T \mathrm{B} \mathrm{d}V$$

$$K^{\mathrm{pp}} = \int_{V\mathrm{el}} \text{-1/K*m} \otimes \mathrm{m} \mathrm{d}V$$

$$m = [\mathrm{m}^1, \mathrm{m}^2, \mathrm{m}^3 ......]$$

$$[\mathrm{m}] = \begin{pmatrix} \mathrm{m}^1, \mathrm{m}^2, \mathrm{m}^3 \ldots \\ \mathrm{m}^1, \mathrm{m}^2, \mathrm{m}^3 \ldots \\ \mathrm{m}^1, \mathrm{m}^2, \mathrm{m}^3 \ldots \\ 0 \ldots\ldots \\ 0 \ldots\ldots \\ 0 \ldots\ldots \end{pmatrix} \text{ for 3D}$$

Solve P inside each element

$$[\mathrm{K}^{uu} - K^{up} (\mathrm{K}^{pp})^{-1} \mathrm{K}^{pu}] \mathrm{u} = F$$

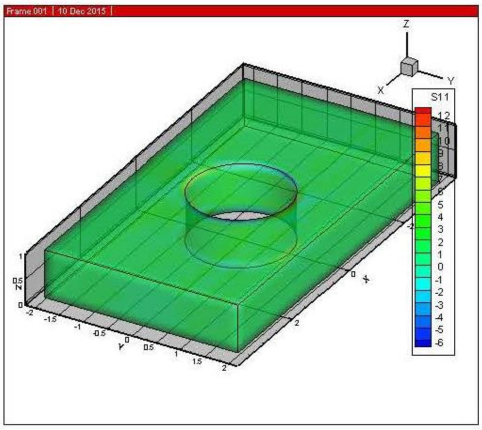
Element stiffness become $[\mathrm{K}^{uu} - K^{up} (\mathrm{K}^{pp})^{-1} \mathrm{K}^{pu}]$, the dof, F residual force can stay the same as usual.

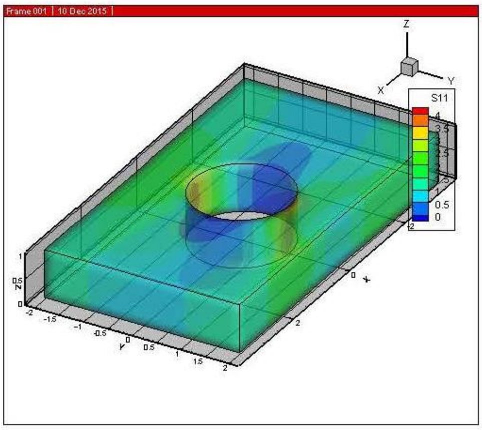
The code is tested by using holeplate input file with passion ration of 0.3 and 0.49999, results are below. It can be seen from the pictures below, hybrid give basically the same results as B bar method. When Possion ratio becomes close to 0.5, basic FEA cannot give correct results anymore, but hybrid element still works. And when possion ratio is 0.3, hybrid element also give similar results as basic FEA.

Possion ratio: 0.49999

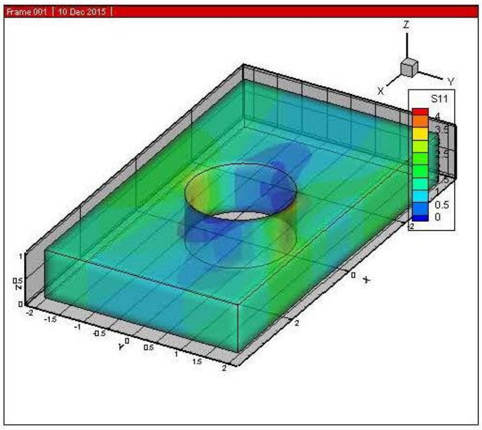
Possion ratio: 0.3

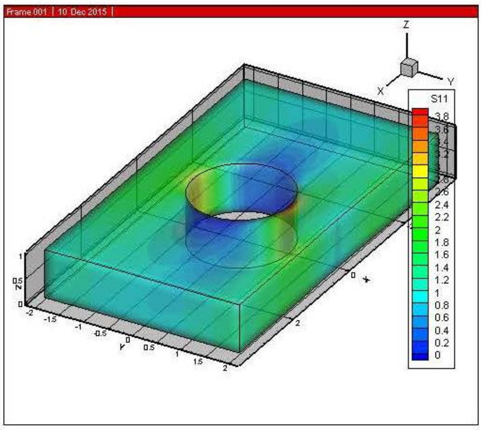
Normal FEA



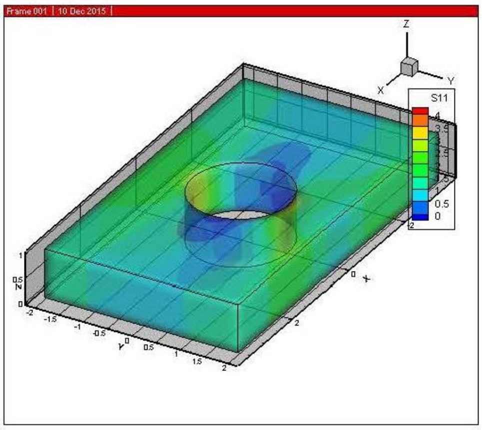


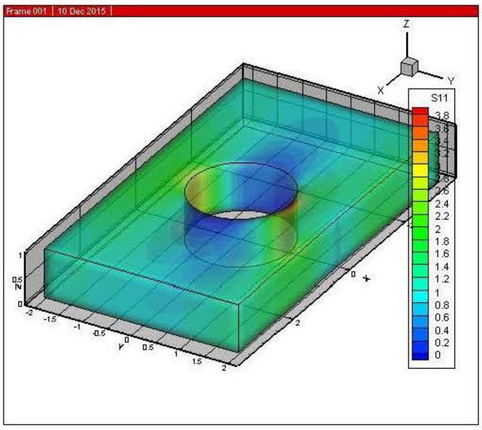
B-bar method:





Hybrid element





Results by hybrid element, B bar and normal FEA

Figure 1

Implement hybrid element by solving P globally:

Tested by using linear elastic input file with only 2 element and there are 8 nodes in each element, one fake node is added to each element to be pressure node. The exact solution at the end of step 3 should be S11=15, S22=S33=S12=S13=S23=0

```
   NODES
    PARAMETERS, 0, 1, 2
   CREATE NODES, 2
  end nodes
%
%   The ELEMENT command defines properties of elements
%   The parameters are no. nodes on the element, total no. state variables, integer ide
  
  ELEMENTS
    PARAMETERS, 9, 90, 1010
%     Define element properties - the values are passed to user subroutine elstif in the
%     For the example provided, the params are Youngs Modulus, Poissons ratio
    PROPERTIES
     100.d0, 0.499999d0
    END PROPERTIES
%   Define mass density (for explicity dynamics only)
%     DENSITY, 10.d0
%   Define element connectivity
%   The element number (first number in the list) is optional, and is ignored in the coo
    CONNECTIVITY, zone1
1,      1, 2, 3, 4, 5, 6, 7, 8,13
2,      2, 9, 10, 3, 6, 11, 12, 7,14
    END CONNECTIVITY
```

Figure 2.Input file

Using user print file to print out displacement and stress at integration points, below is the result at step 3 by using hybrid element and b bar method. It can be seen form the data that b bar method and hybrid element give same results for passion ration 0.499999.

```
    1     1 0.00000D+00 0.00000D+00 0.00000D+00 0.00000D+00 0.00000D+00 0.00000D+00
    1     2 0.11500D+01 0.00000D+00 0.87585D-17 0.15000D+00 0.00000D+00 0.87585D-17
    1     3 0.11500D+01 0.92500D+00 0.45031D-17 0.15000D+00-0.75000D-01 0.45031D-17
    1     4 0.00000D+00 0.92500D+00 0.96256D-17 0.00000D+00-0.75000D-01 0.96256D-17
    1     5 0.00000D+00 0.00000D+00 0.92500D+00 0.00000D+00 0.00000D+00-0.75000D-01
    1     6 0.11500D+01 0.00000D+00 0.92500D+00 0.15000D+00 0.00000D+00-0.75000D-01
    1     7 0.11500D+01 0.92500D+00 0.92500D+00 0.15000D+00-0.75000D-01-0.75000D-01
    1     8 0.00000D+00 0.92500D+00 0.92500D+00 0.00000D+00-0.75000D-01-0.75000D-01
 lmn,kint,x1,x2,x3,s11,s22,s33,s12,s13,s23
    1     1 0.21132D+00 0.21132D+00 0.21132D+00 0.15000D+02-0.88818D-15 0.26645D-14 0.30593D-15-0.76379D-15-0.61516D-15
    1     2 0.78868D+00 0.21132D+00 0.21132D+00 0.15000D+02 0.88818D-15 0.17764D-14 0.10721D-14 0.21274D-14-0.26259D-15
    1     3 0.21132D+00 0.78868D+00 0.21132D+00 0.15000D+02-0.88818D-15 0.88818D-15 0.51879D-15-0.11196D-14 0.39676D-15
    1     4 0.78868D+00 0.78868D+00 0.21132D+00 0.15000D+02 0.00000D+00 0.00000D+00 0.12994D-14-0.10764D-15 0.47467D-15
    1     5 0.21132D+00 0.21132D+00 0.78868D+00 0.15000D+02-0.88818D-15 0.00000D+00-0.43530D-15-0.83045D-15-0.10875D-14
    1     6 0.78868D+00 0.21132D+00 0.78868D+00 0.15000D+02 0.00000D+00-0.88818D-15-0.16496D-14 0.21764D-14-0.64407D-15
    1     7 0.21132D+00 0.78868D+00 0.78868D+00 0.15000D+02 0.00000D+00-0.88818D-15-0.44006D-15-0.12255D-14-0.10447D-15
    1     8 0.78868D+00 0.78868D+00 0.78868D+00 0.15000D+02 0.00000D+00-0.88818D-15-0.15965D-14-0.11181D-16-0.49951D-15
 lmn,node,x1,x2,x3,u1,u2,u3
    2     1 0.11500D+01 0.00000D+00 0.87585D-17 0.15000D+00 0.00000D+00 0.87585D-17
    2     2 0.23000D+01 0.00000D+00 0.80277D-17 0.30000D+00 0.00000D+00 0.80277D-17
    2     3 0.23000D+01 0.92500D+00 0.82104D-17 0.30000D+00-0.75000D-01 0.82104D-17
    2     4 0.11500D+01 0.92500D+00 0.45031D-17 0.15000D+00-0.75000D-01 0.45031D-17
    2     5 0.11500D+01 0.00000D+00 0.92500D+00 0.15000D+00 0.00000D+00-0.75000D-01
    2     6 0.23000D+01 0.00000D+00 0.92500D+00 0.30000D+00 0.00000D+00-0.75000D-01
    2     7 0.23000D+01 0.92500D+00 0.92500D+00 0.30000D+00-0.75000D-01-0.75000D-01
    2     8 0.11500D+01 0.92500D+00 0.92500D+00 0.15000D+00-0.75000D-01-0.75000D-01
 lmn,kint,x1,x2,x3,s11,s22,s33,s12,s13,s23
    2     1 0.12113D+01 0.21132D+00 0.21132D+00 0.15000D+02-0.17764D-14-0.17764D-14 0.11206D-14 0.14707D-14 0.23292D-15
    2     2 0.17887D+01 0.21132D+00 0.21132D+00 0.15000D+02-0.17764D-14-0.88818D-15-0.18040D-15-0.22063D-15 0.97958D-15
    2     3 0.12113D+01 0.78868D+00 0.21132D+00 0.15000D+02 0.00000D+00-0.88818D-15 0.21282D-14 0.17185D-15 0.14618D-15
    2     4 0.17887D+01 0.78868D+00 0.21132D+00 0.15000D+02 0.00000D+00 0.00000D+00 0.27786D-15-0.66660D-15-0.32346D-16
    2     5 0.12113D+01 0.21132D+00 0.78868D+00 0.15000D+02 0.88818D-15 0.00000D+00-0.14280D-14 0.21090D-14-0.14258D-16
    2     6 0.17887D+01 0.21132D+00 0.78868D+00 0.15000D+02-0.88818D-15 0.17764D-14-0.47386D-15-0.49313D-15 0.12738D-14
    2     7 0.12113D+01 0.78868D+00 0.78868D+00 0.15000D+02 0.00000D+00 0.00000D+00-0.22763D-14 0.99727D-15-0.72083D-16
    2     8 0.17887D+01 0.78868D+00 0.78868D+00 0.15000D+02-0.88818D-15 0.88818D-15-0.11777D-14-0.21704D-15 0.59448D-16
```

Figure 3.Results by hybrid element

```
VARIABLES = X,Y,Z,DU1,DU2,DU3,U1,U2,U3,S11,S22,S33,S12,S13,S23
ZONE, T="0.3000D+01" F=FEPOINT, I=   12 J=    2 ET=BRICK
  0.0000000000E+00  0.0000000000E+00  0.0000000000E+00  0.0000000000E+00  0.0000000000E+00  0.0000000000E+00  0.0000000000E+00  0.0000000000E+00
  0.1150000000E+01  0.0000000000E+00 -0.8182278132E-12  0.5000000000E-01  0.0000000000E+00 -0.5515906320E-12  0.1000000000E+00  0.0000000000E+00  -
  0.1150000000E+01  0.9250001500E+00  0.3888713810E-12  0.5000000000E-01 -0.2499995000E-01  0.1165530820E-11  0.1000000000E+00 -0.4999990000E-01  -
  0.0000000000E+00  0.9250001500E+00 -0.7647919965E-13  0.0000000000E+00 -0.2499995000E-01  0.6303934945E-13  0.0000000000E+00 -0.4999990000E-01  -
  0.0000000000E+00  0.0000000000E+00  0.9250001500E+00  0.0000000000E+00  0.0000000000E+00 -0.2499995000E-01  0.0000000000E+00  0.0000000000E+00  -
  0.1150000000E+01  0.0000000000E+00  0.9250001500E+00  0.5000000000E-01  0.0000000000E+00 -0.2499995000E-01  0.1000000000E+00  0.0000000000E+00  -
  0.1150000000E+01  0.9250001500E+00  0.9250001500E+00  0.5000000000E-01 -0.2499995000E-01 -0.2499995000E-01  0.1000000000E+00 -0.4999990000E-01  -
  0.0000000000E+00  0.9250001500E+00  0.9250001500E+00  0.0000000000E+00 -0.2499995000E-01 -0.2499995000E-01  0.0000000000E+00 -0.4999990000E-01  -
  0.2300000000E+01  0.0000000000E+00 -0.1319159043E-11  0.1000000000E+00  0.0000000000E+00 -0.7321505235E-14  0.2000000000E+00  0.0000000000E+00  -
  0.2300000000E+01  0.9250001500E+00 -0.2753637895E-11  0.1000000000E+00 -0.2499995000E-01 -0.2753268263E-11  0.2000000000E+00 -0.4999990000E-01  -
  0.2300000000E+01  0.0000000000E+00  0.9250001500E+00  0.1000000000E+00  0.0000000000E+00 -0.2499995000E-01  0.2000000000E+00  0.0000000000E+00  -
  0.2300000000E+01  0.9250001500E+00  0.9250001500E+00  0.1000000000E+00 -0.2499995000E-01 -0.2499995000E-01  0.2000000000E+00 -0.4999990000E-01  -
    1     2     3     4     5     6     7     8
    2     9    10     3     6    11    12     7

  0.0000000000E+00  0.0000000000E+00  0.1500000002E+02  0.3727498277E-09  0.6468638828E-09  0.1489878547E-16 -0.2991861019E-10 -0.2549269260E-11
  0.0000000000E+00 -0.2666371813E-12  0.1500000002E+02  0.2816149238E-10  0.6756194491E-10  0.1385169950E-10  0.2403779097E-10  0.4023661995E-10
 -0.4999990000E-01 -0.7766594385E-12  0.1500000002E+02 -0.3739420897E-09 -0.2777096071E-09 -0.2253048419E-10  0.4657718512E-10 -0.1957648354E-10
 -0.4999990000E-01 -0.1395185491E-12  0.1500000002E+02 -0.4610536576E-09 -0.8458919360E-09 -0.2489319383E-10  0.4557691906E-10 -0.2028763665E-10
  0.0000000000E+00 -0.4999990000E-01  0.1500000002E+02 -0.1147051307E-08 -0.1095571584E-08 -0.1771975607E-15 -0.4977969403E-10  0.4018501201E-10
  0.0000000000E+00 -0.4999990000E-01  0.1500000002E+02  0.6594953681E-09  0.7658250793E-09  0.5902571076E-10  0.6663445916E-11  0.7340027792E-11
 -0.4999990000E-01 -0.4999990000E-01  0.1500000002E+02  0.3214953905E-09  0.3569323898E-09  0.2951269250E-10  0.5994132153E-10 -0.5247314046E-10
 -0.4999990000E-01 -0.4999990000E-01  0.1500000002E+02  0.3946155878E-09  0.6323100512E-09 -0.9143966415E-10 -0.1030983001E-09  0.2244613008E-10
  0.0000000000E+00 -0.1311837538E-11  0.1500000002E+02  0.7264017345E-10  0.9378057147E-10 -0.4867533519E-15 -0.1405233699E-10 -0.4781609618E-10
 -0.4999990000E-01 -0.3696318818E-15  0.1500000002E+02 -0.8247973675E-09 -0.7015606429E-09 -0.4787096002E-10 -0.1348183763E-09 -0.5181727678E-10
  0.0000000000E+00 -0.4999990000E-01  0.1500000002E+02 -0.1241399681E-08 -0.1185381647E-08 -0.7159606935E-15 -0.2893725017E-10  0.5639269008E-10
 -0.4999990000E-01 -0.4999990000E-01  0.1500000002E+02  0.1638795728E-08  0.1906707507E-08  0.3241387054E-10  0.4058758333E-10  0.5239148133E-10
```

Figure 4.Results by B bar method

Future work

I have only tested hybrid element (solving P globally) with simple 2 element input file and wrote the stress only at intergration points. Next step is to test it with more complicate input file like holepate and use userprint to write out contourplot file, which give the stress at node and can be read by tecplot.