

EN2340 Final Project: Viscoelasticity

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This project consists of two parts. The first part is the FEM implementation of linear viscoelasticity based on standard solid model. And the second part is an extension of the linear model into the finite deformation, based on theory proposed by Reese and Govindjee (1998)

I. Linear viscoelasticity

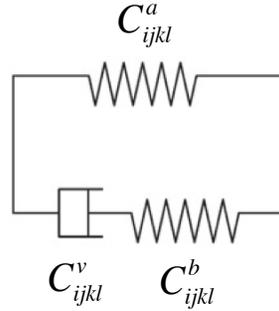


Figure 1: Standard linear solid

In this solid model, two springs and the dashpot are all linear, whose elasticity and viscosity tensors are denoted by C_{ijkl}^a , C_{ijkl}^b and C_{ijkl}^v , giving following constitutive relations:

$$\sigma_{ij}^a = C_{ijkl}^a \varepsilon_{kl}$$

$$\sigma_{ij}^b = C_{ijkl}^b (\varepsilon_{kl} - \varepsilon_{kl}^v) = C_{ijkl}^v \frac{d\varepsilon_{kl}^v}{dt}$$

Combining relations above gives the total constitutive equation of the material:

$$(C^b)_{ijkl}^{-1} \frac{d\sigma_{kl}}{dt} + (C^v)_{ijkl}^{-1} \sigma_{kl} = (C^b)_{ijpq}^{-1} C_{pqkl}^a \frac{d\varepsilon_{kl}}{dt} + \frac{d\varepsilon_{ij}}{dt} + (C^v)_{ijpq}^{-1} C_{pqkl}^a \varepsilon_{kl} \quad (1)$$

Now, write the equation of force balance into weak form:

$$\int \sigma_{ij,j} \delta u_i dV + \int b_i \delta u_i dV = - \int \sigma_{ij} \delta u_{i,j} dV + \int b_i \delta u_i dV = 0, \text{ for any } \delta u_i$$

Inside each element, we use interpolation: $\delta u_i = N^a \delta u_i^a$, which gives the residual:

$$R = \left(- \int \sigma_{ij} N_{,j}^a dV + \int b_i N^a dV \right) \delta u_i^a$$

Then, with θ scheme and given u_i^n , $\Delta u_i = u_i^{n+1} - u_i^n$, σ_{ij}^n and interpolation $u_i = N^a u_i^a$, we can easily compute the increment of stress $\Delta \sigma_{ij} = \sigma_{ij}^{n+1} - \sigma_{ij}^n$ (and therefore the updated stress) at each point of interest from equation 1. And since the updated stresses are obtained, the residual could be calculated simply by plug the stress into the formula at each integration point and do the integral (the procedure for corresponding material tangent would be similar).

The procedure described above is implemented as an ABAQUS UMAT subroutine. To test the validity of the results, we first perform a stress relaxation test. The

geometry of the model is a cube with edge length of 1. For all the three isotropic tensors C_{ijkl}^a , C_{ijkl}^b and C_{ijkl}^v , the shear modulus/viscosity is taken as 1, and the bulk part is 5. The corresponding stress relaxation curve versus time is given in Fig. 2.

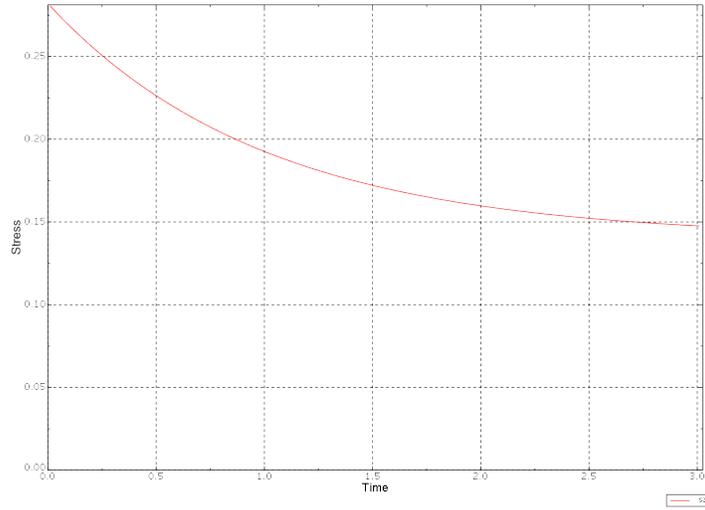


Figure 2: Stress relaxation with linear viscoelastic model

This result is consistent with the classic theory of exponential decay of stress, which predicts $\sigma = 0.14 + 0.14 \cdot \exp(-t)$.

Then, with the same model and material parameters, we applied a cyclic uniaxial stretch. In this case, one cube surface is given an displacement of $0.05\sin(2\pi t)$, corresponding to a sinusoidal strain with magnitude of 0.05, which starts from $t=0$. And the total time period computed is 10. The hysteresis curve is shown in Fig. 3

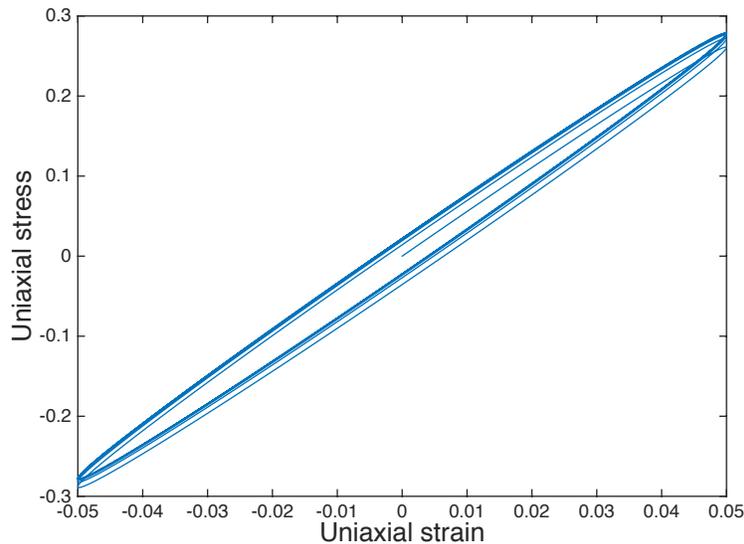


Figure 3: Hysteresis of linear viscoelastic material

It can be observed after several cycles a stable elliptic hysteresis is developed, which is also consistent with the theory.

II. Finite deformation viscoelasticity

Reese and Govindjee (1998) proposed a theory of finite deformation viscoelasticity as an extension of the classical standard linear solid model. In his model, the deformation gradient is decomposed into $\mathbf{F}^t = \mathbf{F}^e \mathbf{F}^i$, where \mathbf{F}^e corresponds to the deformation of the dashpot in the standard solid model (Fig. 4) It should be noted that in our following part of the report, for simplicity the upper note 't' is sometimes omitted for the total deformation gradient, but would stay for the stress on the upper spring in Fig.4.

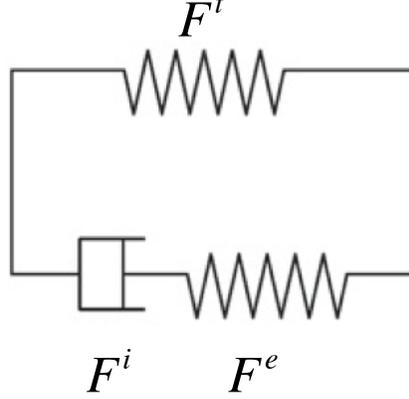


Figure 4: Reese - Govindjee model

The authors proposed that, besides from the non-linear constitutive equations that govern two springs in the model, another governing equation is :

$$-\frac{1}{2} \mathcal{L}_v \mathbf{B}^e \cdot (\mathbf{B}^e)^{-1} = \mathcal{V}^{-1} : \boldsymbol{\sigma}^e$$

where $\mathcal{L}_v \mathbf{B}^e$ represents the Lie derivative of the non-equilibrium left Cauchy-Green tensor $\mathbf{B}^e = \mathbf{F}^e \cdot (\mathbf{F}^e)^T$. And \mathcal{V} is a fourth order isotropic viscosity tensor (the same as the C_{ijkl}^v we used in our linear viscoelastic model). This equation gives the time-dependent behavior of the material.

Putting this model into principle of virtual work, together with the interpolation of displacement and test function in each element, we could easily get the weak form for finite element formulation:

$$R = \int \sigma_{ij} \frac{\partial N^a}{\partial y_j} \delta u_i^a dV - \int b_i N_{,j}^a \delta u_i^a dV,$$

$$\Delta R = \int \frac{\partial J \sigma_{ij}}{\partial F_{kl}} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial y_j} \Delta u_k^b \delta u_i^a dV_0 - \int \Delta b_i N_{,j}^a \delta u_i^a dV$$

where J is the determinant of the total deformation gradient. Notice the Kirchhoff stress $J \sigma_{ij} = J \sigma_{ij}^t + J^e \sigma_{ij}^e \cdot (J/J^e)$ (J^e is the determinant of \mathbf{F}^e), the above formula can be rewritten as

$$\Delta R = \int \frac{\partial J \sigma_{ij}^t}{\partial B_{mn}} \frac{\partial B_{mn}}{\partial F_{kl}} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial y_j} \Delta u_k^b \delta u_i^a dV_0 - \int b_i N_{,j}^a \delta u_i^a dV$$

$$\begin{aligned}
& + \int \frac{J}{J^e} \frac{\partial J^e \sigma_{ij}^e}{\partial B_{mn}^e} \frac{\partial B_{mn}^e}{\partial F_{kl}} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial y_j} \Delta u_k^b \delta u_i^a dV_0 \\
& + \int \frac{\sigma_{ij}^e}{J^e} \frac{\partial J}{\partial F_{kl}} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial y_j} \Delta u_k^b \delta u_i^a dV_0 \\
& - \int \frac{\sigma_{ij}^e J}{(J^e)^2} \frac{\partial J^e}{\partial B_{mn}^e} \frac{\partial B_{mn}^e}{\partial F_{kl}} \frac{\partial N^b}{\partial x_l} \frac{\partial N^a}{\partial y_j} \Delta u_k^b \delta u_i^a dV_0
\end{aligned}$$

In the formula above, all the Kirchhoff stresses' derivative with respect to corresponding left Cauchy-Green tensor could be easily obtained through material's elastic constitutive relations. And since J is the determinant of deformation gradient, and J^e is the square root of \mathbf{B}^e 's determinant, all the derivatives about them can also be easily computed. The only term left to obtain is $\frac{\partial B_{mn}^e}{\partial F_{kl}}$.

Reese and Govindjee (1998) suggest that, the time evolution equation:

$$-\frac{1}{2} \mathcal{L}_v \mathbf{B}^e \cdot (\mathbf{B}^e)^{-1} = \mathcal{V}^{-1} : \boldsymbol{\sigma}^e$$

can be numerically solved with the trial step technique. At the beginning of each time step t_n , we compute a trial value of $\mathbf{B}^e = \mathbf{B}^{trial} = \mathbf{F}_{t=t_n} \cdot (\mathbf{C}^i)_{t=t_{n-1}}^{-1} \cdot \mathbf{F}_{t=t_n}^T$, where $\mathbf{C}^i = (\mathbf{F}^i)^T \mathbf{F}^i$. And then, the real \mathbf{B}^e is obtained by solving the non-linear equation with Newton iteration:

$$\mathbf{B}_{t=t_n}^e = \exp[-2\Delta t (\mathcal{V}^{-1} : \boldsymbol{\sigma}^e)_{t=t_n}] \cdot \mathbf{B}^{trial}$$

With the equation above, actually we could obtain not only the \mathbf{B}^e at each step, but also $\frac{\partial B_{mn}^e}{\partial F_{kl}}$. By taking the derivative of F_{kl} on two sides, we get an equation:

$$\begin{aligned}
\left[\delta_{ik} \delta_{jl} + 2\Delta t \mathcal{V}_{ipmn}^{-1} \frac{\partial \sigma_{mn}}{\partial B_{kl}^e} B_{pj}^e \right] \frac{\partial B_{kl}^e}{\partial F_{st}} = \exp[-2\Delta t (\mathcal{V}^{-1} : \boldsymbol{\sigma}^e)_{t=t_n}]_{is} (C_{t=t_{n-1}}^i)_{tp}^{-1} (F_{t=t_n}^T)_{pj} \\
+ \exp[-2\Delta t (\mathcal{V}^{-1} : \boldsymbol{\sigma}^e)_{t=t_n}]_{ip} (F_{t=t_n})_{pq} (C_{t=t_{n-1}}^i)_{qt}^{-1} \delta_{js}
\end{aligned}$$

Solving it, we get the $\frac{\partial B_{mn}^e}{\partial F_{kl}}$, and therefore every term in the weak form can be computed.

Implementing this model as an ABAQUS UEL, we select the neo-Hookean model for the springs in the model, and both shear moduli are 1, and bulk moduli are 5. Similarly, the shear viscosity for \mathcal{V} is also 1 and bulk viscosity is 5. We use the same geometry and mesh in our linear viscoelastic part, and also put it under a uniaxial stress relaxation test. Again, the applied step strain is 0.05. The result is plotted in Fig. 5

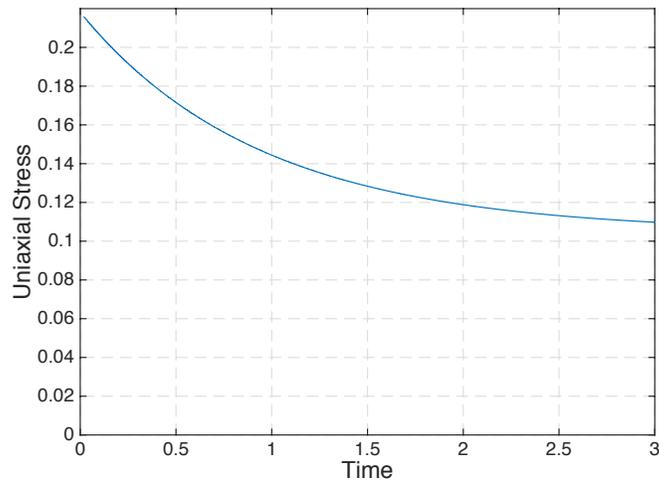


Figure 5: Stress relaxation of Reese and Govindjee model

From Fig. 5, we can see, during the relaxation, the stress level would gradually decrease to half of initial value, which is consistent with the model (Fig. 4), and the characteristic time of the decay is also around 1. However, it should be noted that the initial level of the stress is somewhat lower than the theoretical value, which indicates our code still have problems to fix.

In conclusion, in this project we first implement the linear viscoelastic material model in UMAT, which successfully reproduces the theoretical results. And then we attemptively implement the finite deformation viscoelastic model proposed by Reese and Govindjee (1998), whose result, although partially consistent with the model, indicates further fixes are necessary for our code.