EN2340 Final Project: Continuum-Based Beam Element

Zhi Li

Gihub Link: https://github.com/zhili93/EN234_FEA.git

Abstract: Two nodes 2D continuum-based beam element is implemented in EN234_FEA codes. Several tests have been run to verify this beam element.

1 Background

Structural element, such as beam element, shell element, are widely used in engineering practice. Compared with continuum element, those structural elements have the advantages of lower computational costs and higher stability. There are generally two different methods to develop structural elements[1]. One is developing directly from the weak form of classical beam/shell equations of equilibrium/ momentum. This method is difficult due to the complexity of governing equations of structural elements. The other method is developing the element by degenerating from a corresponding continuum element with some structural assumptions. A 2D continuum-based beam element is developed below to show the general ideas.

2 Continuum-based beam element

Figure 1 shows the sketch of a two nodes 2D beam element (5, 6) which is degenerated from a four nodes 2D quadratic element (1, 2, 3, and 4). Nodes 5, 6 are called master nodes, and nodes 1,2,3,4 are called slave nodes whose coordinates and degrees of freedom could be represented in terms of the corresponding terms of master nodes.

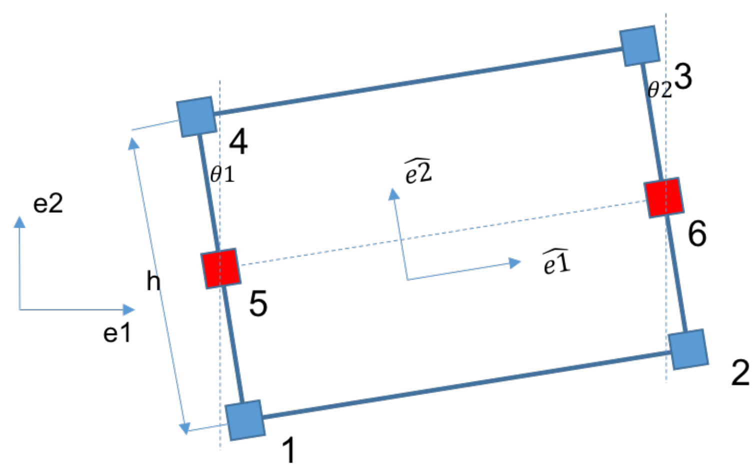

Figure 1 sketch of continuum-based beam element

Master nodes have three degrees of freedom$\{U1, U2, \theta\}$. The coordinates and displacements of slave nodes (1, 4) can be computed from master node 5 under Timoshenko assumption:

$$coordinate: \qquad \begin{cases} x_1^4 = x_1^5 - 0.5 * h * \sin(\theta 1) \\ x_2^4 = x_2^5 + 0.5 * h * \cos(\theta 1) \\ x_1^1 = x_1^5 + 0.5 * h * \sin(\theta 1) \\ x_2^1 = x_2^5 - 0.5 * h * \cos(\theta 1) \end{cases} \tag{1}$$

$$displacement: \qquad \begin{cases} \underline{dU^4} = \underline{dU^5} + d\theta 1\, \underline{e3} \times (\underline{x^4} - \underline{x^5}) \\ \underline{dU^1} = \underline{dU^5} + d\theta 1\, \underline{e3} \times (\underline{x^1} - \underline{x^5}) \end{cases} \tag{2}$$

The linear transformation that link DOFs of beam element with DOFs of solid element is:

$$\underline{U^s} = T\underline{U^m} \tag{3}$$

Where,

$$\underline{U^s} = \{U_1^1, U_2^1, U_1^2, U_2^2, U_1^3, U_2^3, U_1^4, U_2^4\}$$

$$\underline{U^m} = \{U_1^5, U_2^5, \theta 1, U_1^6, U_2^6, \theta 2\}$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ x_2^5 - x_2^1 & x_1^1 - x_1^5 & 0 & 0 & 0 & 0 & x_2^5 - x_2^4 & x_1^4 - x_1^5 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & x_2^6 - x_2^2 & x_1^6 - x_1^2 & x_2^6 - x_2^3 & x_1^6 - x_1^3 & 0 & 0 \end{pmatrix}^T \tag{4}$$

Then we have:

$$\underline{\varepsilon} = BT\underline{U^m} \tag{5}$$

Where B is the standard B matrix for the four nodes 2D element.

The calculation of stress is performed in corotational coordinates $\{\underline{\widehat{e1}}, \underline{\widehat{e2}}, \underline{\widehat{e3}}\}$, where

$$\underline{\widehat{e1}} = \frac{\partial x}{\partial \xi} \Big/ \left|\frac{\partial x}{\partial \xi}\right| \quad \underline{\widehat{e3}} = \underline{e3} \quad \underline{\widehat{e2}} = \underline{\widehat{e3}} \times \underline{\widehat{e1}} \tag{6}$$

We adopted the so-called plane stress condition or zero normal stress condition, which says the transverse normal stress $\widehat{\sigma_{yy}}$ is negligible. So we have the following constitutive relation:

$$\underline{\hat{\sigma}} = D\underline{\hat{\varepsilon}} \tag{7}$$

Where,

$$\underline{\hat{\sigma}} = \begin{pmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{12} \end{pmatrix} \quad \underline{\hat{\varepsilon}} = \begin{pmatrix} \hat{\varepsilon}_{11} \\ 2\hat{\varepsilon}_{12} \end{pmatrix} = R \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix} \quad D = \begin{pmatrix} E & 0 \\ 0 & E/2(1+\upsilon) \end{pmatrix}$$

$$R = \begin{pmatrix} \left(\underline{\widehat{e1}} * e1\right)^2 & \left(\underline{\widehat{e2}} * e1\right)^2 & \left(\underline{\widehat{e1}} * e1\right) * \left(\underline{\widehat{e1}} * e2\right) \\ \left(\underline{\widehat{e1}} * e1\right)\left(\underline{\widehat{e2}} * e1\right) & \left(\underline{\widehat{e1}} * e2\right)\left(\underline{\widehat{e2}} * e2\right) & \left(\underline{\widehat{e1}} * e1\right)\left(\underline{\widehat{e2}} * e2\right) + \left(\underline{\widehat{e1}} * e2\right)\left(\underline{\widehat{e2}} * e1\right) \end{pmatrix}$$

Combining (5) (7), we get $\underline{\widehat{\sigma}} = DRBT\underline{U^m}$.

Recalling principle of virtual work equation for a standard four nodes 2D element:

$$\int C_{ijkl} \frac{\partial N^a}{\partial x_j} \frac{\partial N^b}{\partial x_l} U_l^{s\,b} dv = -\int \sigma_{ij}^0 \frac{\partial N^a}{\partial x_j} dv + external\ forcing \tag{8}$$

Substituting (3) (5) (7) into (8), we get the finite element scheme for 2D beam element:

$$\sum_{element} [k_{el}]\underline{U^m} = \sum_{element} r_{el} + forcing$$

$$[k_{el}] = \int (RBT)^T DBRBT dv \qquad r_{el} = -\int (RBT)^T \underline{\hat{\sigma}} dv$$

In order to avoid shear locking, the integral is calculated using ''Trapezoidal Rule'' with 5 points in the line of $\xi = 0$.

3 Examples

3.1 Shear Lock and Unlock

Figure 2 and figure 3 respectively show the FEA result for a beam fixed at one end and loaded at the other end. The dimension of the beam is 33 units long with a constant cross section of 1*1. In figure 2, standard 4 nodes Gauss interpolation points are used, while in figure 3 Trapezoidal Rule with 5 points in $\xi = 0$ is adopted. From these figures, we could see that for this relative long beam, Gauss interpolation will result in shear lock, while trapezoidal rule could solve this problem. Figure 4 and Figure 5 shows the FEA results of a relatively short beam with a cross section of 8*8. Under this condition, no shear lock happens.

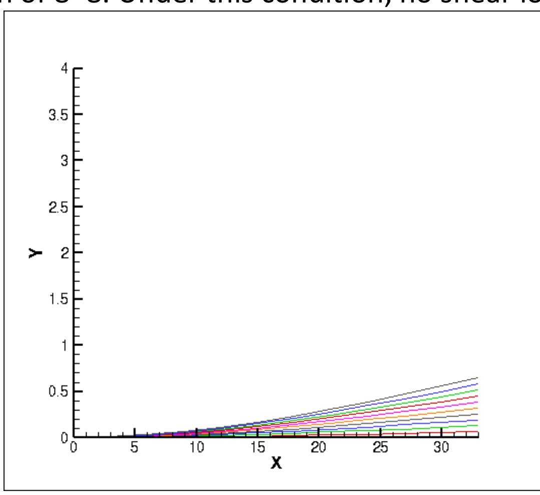

Figure 2 Gauss interpolation (long beam)

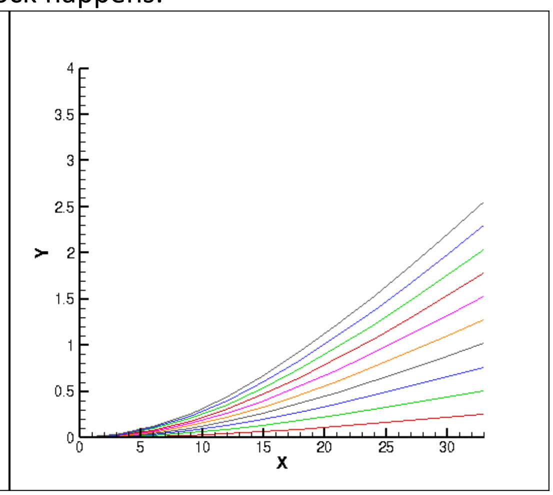

Figure 3 Trapezoidal Rule (long beam)

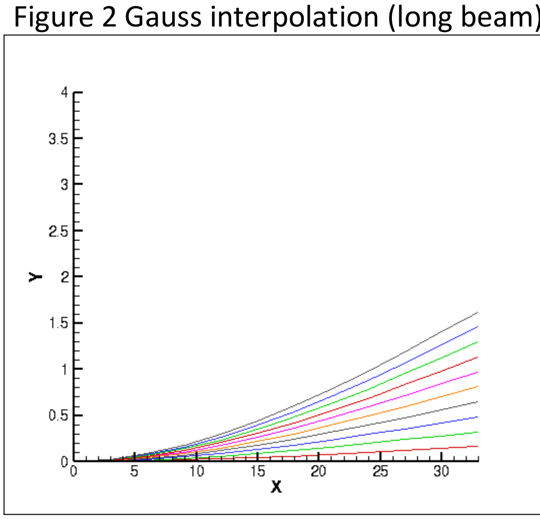

Figure 4 Gauss interpolation (short beam)

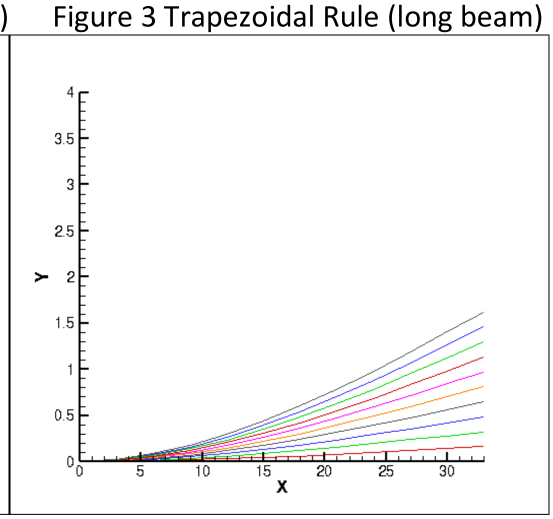

Figure 5 Trapezoidal Rule (short beam)

3.2 Comparison of Timoshenko beam and Euler beam

Timoshenko beam and Euler beam differ in the structural assumptions that Euler theory assumes that the plane normal to the midline remain plane and normal, while Timoshenko beam theory says plane normal to the midline remain plane but no longer normal. Figure 6 shows the different kinematic assumptions of two different beam theories. The continuum-based beam element adopts Timoshenko structural assumptions which count the effects of shear force. For a cantilever beam fixed at one end and subjected to pin load at free end, these two beam theories give different deflection equations:

$$Euler\ beam: \quad w(x) = \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \tag{9}$$

$$Timoshenko\ beam: \quad w(x) = \frac{P}{EI}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - \frac{p}{KAG}\left(xl - \frac{x^2}{2}\right) \tag{10}$$

Figure 7 shows the results of Timoshenko beam theory, Euler beam theory and FEA for the same cantilever beam model. The dimension of this beam is 33*6*6. From the figure, we could see that the FEA coincides with Timoshenko theory.

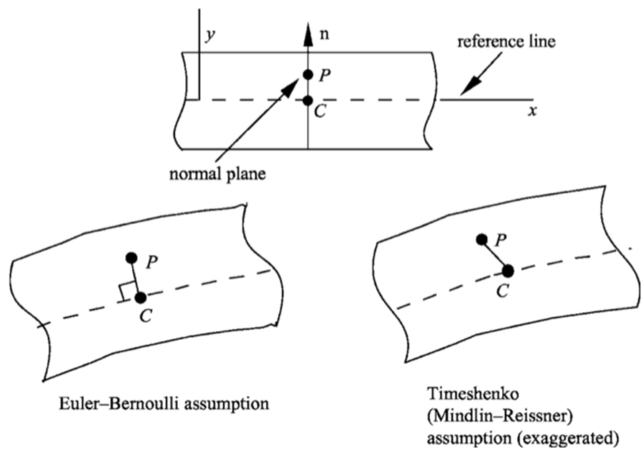

Figure 6 Euler beam and Timoshenko beam

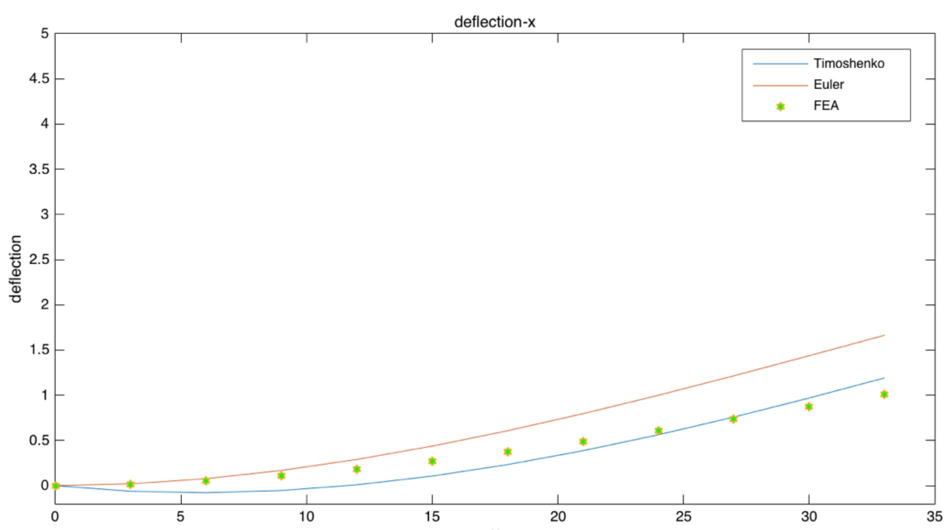

Figure 7 deflection –x

3.3 distributed load on simply supported beam

Figure 8 shows the FEA results for a simply supported beam under constant distributed load. The deflection is calculated through Timoshenko theory as following:

$$w(x) = \frac{q}{24EI}(-2Lx^3 + x^4 + L^3x) + \frac{q}{KAG}\left(-\frac{xl}{2} + \frac{x^2}{2}\right) \tag{11}$$

Figure 9 shows the comparison between Timoshenko results and FEA results. The dimension of the beam is 33*6*6.

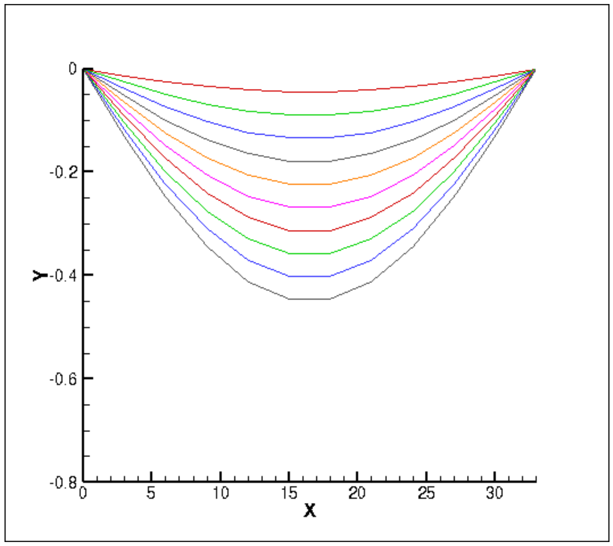

Figure 8 deflection of simply supported beam

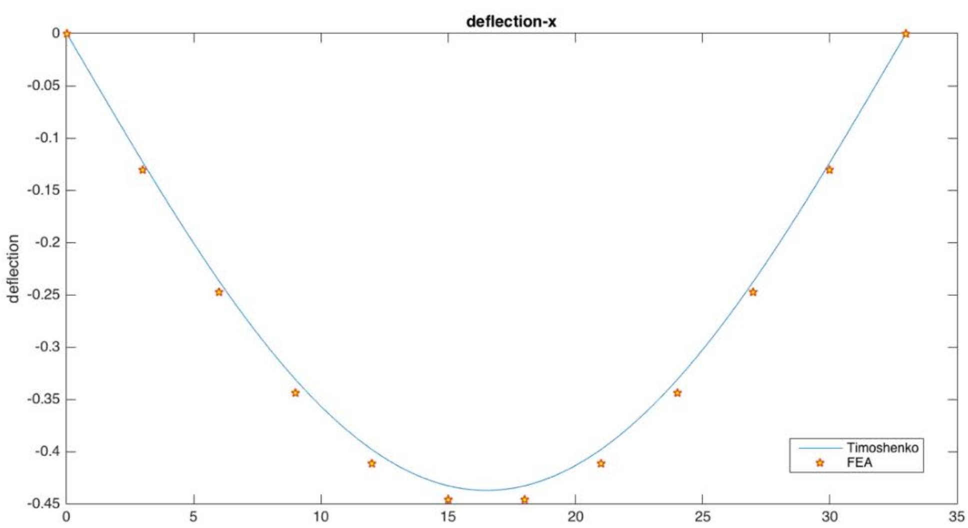

Figure 9 comparison between Timoshenko beam theory with FEA result

Reference

[1] T. Belytschko, W. K. Liu, B. Moran, and K. Elkhodary, *Nonlinear finite elements for continua and structures*: John Wiley & Sons, 2013.