Project Report: ENGN 2340
Thermoelastic coupling: A simple problem

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Objective

To understand the thermoelastic coupling by solving a simplified problem.

Formulation

Simplifications:

- 1D linear elastic (small strain) bar is considered
- Uniaxial stress

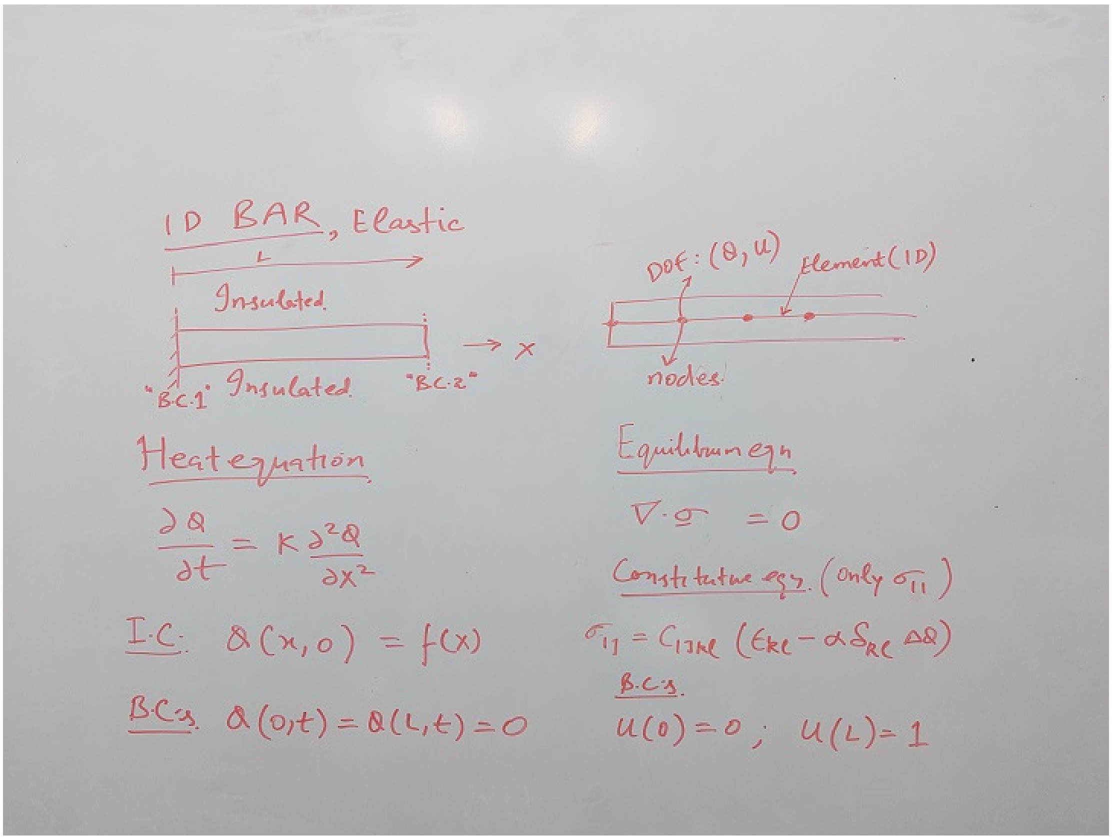


Figure 1: 1D elastic bar formulation

Note: The heat equation poses IV-BVP. Equilibrium equation poses BVP.

Discretization

The governing PDEs are discretized using galerkin method. The strong form is multiplied by a weight function and weak form is obtained.

$$\int_R \left(\frac{\partial \theta}{\partial t} - k \frac{\partial^2 \theta}{\partial x^2} + Q \right) \eta_{(1)} = 0 \tag{1}$$

Integrating by parts, and assuming η is zero where temperature field in prescribed

$$\int_R \left(\frac{\partial \theta}{\partial t} \eta_{(1)} + k \frac{\partial \theta}{\partial x} \frac{\partial \eta_{(1)}}{\partial x} + Q \eta_{(1)} \right) \tag{2}$$

Using interpolation functions as

$$\eta = N^a \eta_{(1)}^a, \quad \theta = N^b \theta^b \tag{3}$$

$$\int_R N^a \left(N^b \frac{\partial \theta_b}{\partial t} + k \frac{\partial N^a}{\partial x} \frac{\partial N^b}{\partial x} \theta_b Q \right) \eta^a \tag{4}$$

The matrix form of the above equation can be written as

$$M^{ab} \frac{\partial \theta^b}{\partial t} + K^{ab} \theta^b = F \tag{5}$$

where

$$M^{ab} = \int_R N^a N^b, \quad K^{ab} = \int_R \frac{\partial N^a}{\partial x} \frac{\partial N^b}{\partial x}, \quad F = \int_R N^a Q \tag{6}$$

Now, using time integrating scheme (from class notes)

$$M^{ab} \frac{\Delta \theta^b}{\Delta t} + K^{ab} \left(\theta^b (1 - \beta) + (\theta^b + \Delta \theta^b) \beta \right) = F \tag{7}$$

Note: The above equation does not depend on the stress field, therefore, can be solve independently to compute $\Delta \theta^b$.

Thermoelastic coupling

The equilibrium equation and the constitutive relation are given by

$$\sigma_{,x} = 0 \tag{8}$$

and

$$\sigma = E(u_{,x} - \alpha \theta) \tag{9}$$

respectively, where α is the coupling parameter. Substituting the above equation in the (8), we can write the weak form as

$$\int_R \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} - \alpha \Delta \theta \right) \eta_{(2)} = 0 \tag{10}$$

Integration by parts

$$-\int_R \left(\frac{\partial u}{\partial x} - \alpha\theta_{avg}\right)\frac{\partial \eta_{(2)}}{\partial x} = 0, \tag{11}$$

where θ_{avg} is the average over the element. Using the interpolation functions, the matrix form can be written as follows

$$K^{ab}u^b = F^a \tag{12}$$

where

$$K^{ab} = \int_R \frac{\partial N^a}{\partial x}\frac{\partial N^b}{\partial x}, \quad F^a = \alpha\,\theta_{avg}\int_R \frac{\partial N^a}{\partial x} \tag{13}$$

Algorithm

Step 1: Solve equation(7) and get $\Delta\theta$

Step 2: Find θ_{avg} which in the element is given by (@ integration point)

$$\theta_{avg} = N^1\theta^1 + N^2\theta^2 \tag{14}$$

Step 3: Solve equation (12) for the displacement field

Approach 2

This problem can also be formulated as the coupled problem in the following manner

$$\int_R \begin{bmatrix} \eta_{(1)} & \frac{\partial \eta_{(1)}}{\partial x} & \frac{\partial \eta_{(2)}}{\partial x} \end{bmatrix} \begin{bmatrix} \frac{\partial \theta}{\partial t} \\ \kappa\frac{\partial \theta}{\partial x} \\ \sigma \end{bmatrix} = 0 \tag{15}$$

So, essentially a system of equations denoted by $R(u + \Delta u) = 0$. Using taylor expansion, we can compute Δu. The equation becomes

$$K\Delta U = -R(U) \tag{16}$$

where $[K]$ and $R(U)$ are usual FEA matrices given by $B^T DB$ and $B^T\ \bar{\sigma}$ (column matrix in equation (15)). For this problem, the $[B]$ and the $[D]$ matrices are given as

$$B = \begin{bmatrix} N^1 & 0 & N^2 & 0 \\ \frac{\partial N^1}{\partial x} & 0 & \frac{\partial N^2}{\partial x} & 0 \\ 0 & \frac{\partial N^1}{\partial x} & 0 & \frac{\partial N^2}{\partial x} \end{bmatrix}$$

and

$$D = \begin{bmatrix} \frac{1}{\Delta t} & 0 & 0 \\ 0 & K & 0 \\ -E\alpha & 0 & E \end{bmatrix}$$

Algorithm

Step 1: Use the initial condition vector U to find $R = -B^T DBU$

Step 2: Get dU using the relation $KdU = R$. For this, we may use newton iterations which are

$dU \to W + dW$

Solve for K dW = -R(U+W)

Update W

Step 3: Update U and R

Validating the code

It can be seen from the equations that the problem is semi-coupled. This means that the heat equation can be solved independently, and the result can be used to determine the dependent stress field. However, it is important to note that the time scales associated with the stress wave and the heat wave are very different. As a consequence, we ignore the inertia term in the equilibrium equation. In order to validate the code, the 1D heat transfer equation has been studied.

Test example: 1D bar with $L = 1$, $\kappa = 1$, $\theta(0,t) = 0$, $\theta(1,t) = 0$, $Q = 1$ and $\theta(x,0) = \sin(\pi x)$

Analytical solution

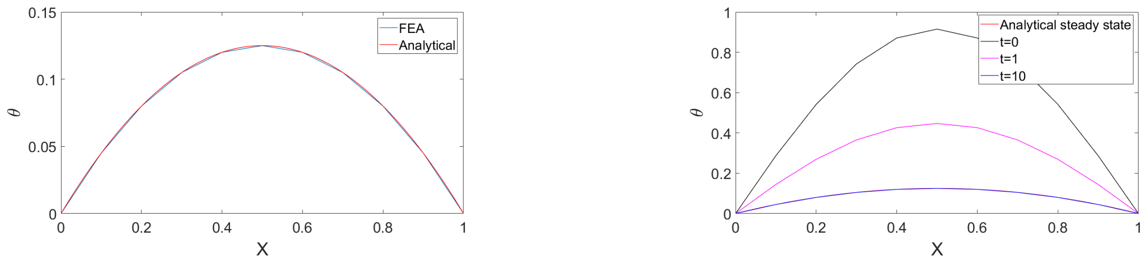
The steady state and unsteady solution for the given set of boundary conditions is given by

$$\theta(x) = -Qx^2/2 + x/2, \tag{17}$$

and

$$\theta(x,t) = \exp(-\kappa\pi^2 t)\sin(\pi x) - Qx^2/2 + x/2, \tag{18}$$

respectively. The FEA solution is compared with the above analytical expressions as follows



(a) Steady state 1D heat transfer.

(b) Unsteady 1D heat transfer.

Figure 2: Comparison of FEA and analytical solution in the case of 1D heat transfer. In this particular example, interpolation functions are taken to be linear, and the total number of elements are 10.

Thermoelastic problem

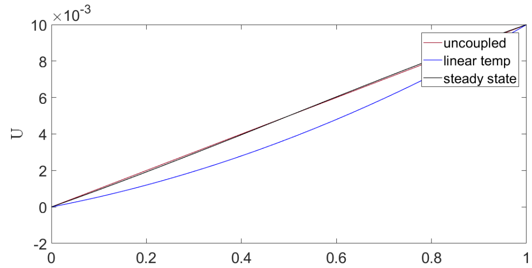
Test example: 1D bar with $L = 1$, $\sigma(x) = E(\epsilon(x) - \alpha\theta(x))$ and $\alpha = 0.01$. The following cases are studied:

- Uncoupled: $\alpha = 0$
- Linear: $\theta(x) = x$
- Steady state: $\theta(x) = -x^2/2 + x/2$

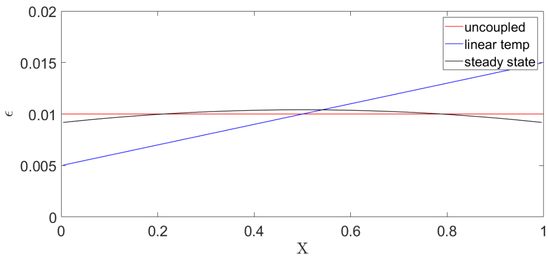
Displacement B.C

B.Cs: $U(0) = 0$ and $U(1) = 0.01$.

Guess: Since $\nabla \cdot \sigma = 0$, the stress should come out to be constant.



(a) Displacement variation. The analytical solutions are; uncoupled: $U(x) = 0.01x$, linear temp: $U(x) = 0.005x + 0.005x^2$ and steady state: $U(x) = 0.00916667x + 0.0025x^2 - 0.00166667x^3$.



(b) Strain variation. The analytical expression are; uncoupled: $\epsilon(x) = 0.01$, linear temp: $\epsilon(x) = 0.005 + 0.01x$ and steady state: $\epsilon(x) = 0.00916667 + 0.005x - 0.005x^2$.

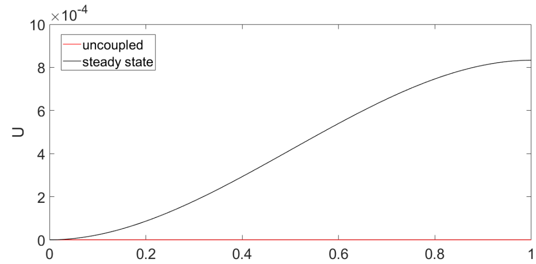
Figure 3: In this example, interpolation functions are taken to be linear, and the total number of elements are 120. Note: The displacements are evaluated at the nodes, but the strains are evaluated at the integration points.

Remark: The FEA solution perfectly matches the analytical solution.

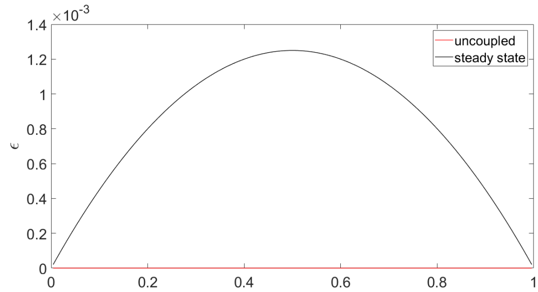
Traction free B.C

B.Cs: $U(0) = 0$ and $\sigma(1) = 0$.

Guess: Since one of the end is free to expand, the bar should again attain the stress-free state.



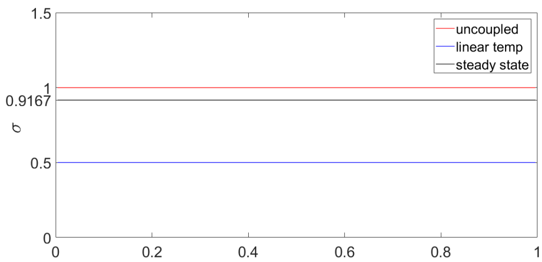
(a) Displacement variation. The analytical solutions are; uncoupled: $U(x) = 0$ and steady state: $U(x) = x(-8.67362 * 10^{-19} + 0.0025x - 0.00166667x^2)$



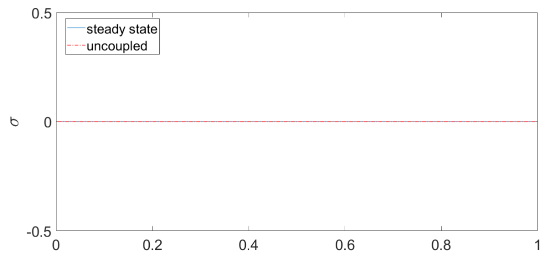
(b) Strain variation. The analytical solutions are; uncoupled: $\epsilon(x) = 0$ and steady state: $\epsilon(x) = 8.67362 * 10^{-19} + 0.005x - 0.005x^2$

Figure 4: In this example, interpolation functions are taken to be linear, and the total number of elements are 120. Note: The displacements are evaluated at the nodes, but the strains are evaluated at the integration points.

Remark: The FEA solution perfectly matches the analytical solution.



(a) B.C: Prescribed displacement



(b) B.C: Traction free

Figure 5: In this example, interpolation functions are taken to be linear, and the total number of elements are 120. Note: The stresses are evaluated at the integration points.

Remark: The FEA solution perfectly matches the analytical solution.

Future directions

Finite deformations or plastic deformations can be incorporated in the present formulation.