B-bar element in hyperelasticity

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1 Problem and b-bar method

Since a lot of hyperelastic materials have much larger bulk modulus than their shear modulus, we can consider them incompressible, so it is not feasible to use traditional elements. Here b-bar element can be used to predict their mechanical behavior.

To begin with, the virtual work equation is

$$\int_{V_0} \tau_{ij}[\bar{F}_{kl}]\delta\bar{L}_{ij}dV_0 - \int_{V0} \rho_0 b_i \delta v_i dV_0 - \int_{\partial V_0} t_i^* \delta v_i \eta dA_0 = 0$$

where $\bar{F}$ is modified deformation gradient and τ_{ij} is Kirchhoff stress tensor and η is volume averaged Jacobian. For this type of element, the stiffness matrix K can be expressed as

$$\mathbf{K} = \int_{\mathbf{V_0}} \bar{\mathbf{B}}^{\mathbf{T}}\mathbf{DGB}^* \mathrm{dV}_0 + \int_{\mathrm{V}_0} -\mathbf{\Sigma} + \frac{\tau_{\mathbf{nn}}}{\mathrm{n}}(\mathbf{P}+\mathbf{Q})\mathrm{dV}_0$$

where bold letter denotes matrix that need to be implemented,

$$P_{aibk} = \frac{1}{\eta V_{el}} \int_{V_{el}} (J\frac{\partial N^b}{\partial y_k}\frac{\partial N^a}{\partial y_i} - J\frac{\partial N^a}{\partial y_k}\frac{\partial N^b}{\partial y_i})dV - \overline{\frac{\partial N^a}{\partial y_i}}\,\overline{\frac{\partial N^b}{\partial y_k}}$$
$$Q_{aibk} = \frac{\partial N^a}{\partial y_k}\frac{\partial N^b}{\partial y_i}$$
$$\Sigma_{aibk} = \frac{\partial N^a}{\partial y_k}\tau_{ij}[\bar{F}_{kl}]\frac{\partial N^b}{\partial y_j}$$

D matrix, which originally maps strain to stress, also needs to be rewrite

$$D = \frac{\mu}{J^{2/3}} \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1/2 & & \\ & & & & 1/2 & \\ 0 & & & & & 1 \end{bmatrix} + \frac{\mu}{3J^{2/3}}(\frac{B_{nn}}{3}\underline{I}\otimes\underline{B}^{-1} - \underline{I}\otimes\underline{I} - \underline{B}\otimes\underline{B}^{-1})$$
$$+ KJ(J-1/2)\underline{I}\otimes\underline{B}^{-1}$$

G matrix just represents the derivative of B over F, which is

$$G = \begin{bmatrix} 2B_{11} & 0 & 0 & 2B_{12} & 0 & 2B_{13} & 0 & 0 & 0 \\ 0 & B_{22} & 0 & 0 & B_{12} & 0 & 0 & B_{23} & 0 \\ 0 & 0 & 2B_{33} & 0 & 0 & 0 & 2B_{13} & 0 & 2B_{23} \\ 2B_{12} & B_{12} & 0 & 2B_{22} & B_{11} & 2B_{23} & 0 & B_{13} & 0 \\ 2B_{13} & 0 & 2B_{13} & 2B_{23} & 0 & 2B_{33} & 2B_{11} & 0 & 2B_{12} \\ 0 & B_{23} & 2B_{23} & 0 & B_{13} & 0 & 2B_{12} & B_{33} & 2B_{22} \end{bmatrix}$$

B-bar element is to add a simple matrix to the original B matrix

$$\bar{B} = B + \begin{bmatrix} \overline{\frac{\partial N^1}{\partial y_1}} - \frac{\partial N^1}{\partial y_1} & \overline{\frac{\partial N^1}{\partial y_2}} - \frac{\partial N^1}{\partial y_2} & \overline{\frac{\partial N^1}{\partial y_3}} - \frac{\partial N^1}{\partial y_3} & \dots \\ \overline{\frac{\partial N^1}{\partial y_1}} - \frac{\partial N^1}{\partial y_1} & \overline{\frac{\partial N^1}{\partial y_2}} - \frac{\partial N^1}{\partial y_2} & \overline{\frac{\partial N^1}{\partial y_3}} - \frac{\partial N^1}{\partial y_3} & \dots \\ \overline{\frac{\partial N^1}{\partial y_1}} - \frac{\partial N^1}{\partial y_1} & \overline{\frac{\partial N^1}{\partial y_2}} - \frac{\partial N^1}{\partial y_2} & \overline{\frac{\partial N^1}{\partial y_3}} - \frac{\partial N^1}{\partial y_3} & \dots \\ 0 & 0 & 0 & \dots \\ \dots & & & \end{bmatrix}$$

RHS vector is

$$\mathbf{R} = \int_{\mathbf{V_0}} \mathbf{\bar{B}^T} \sigma \mathrm{dV_0}$$

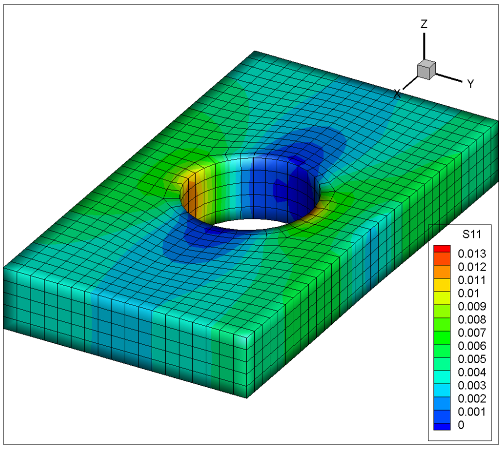
where the σ here actually stores the Kirchhoff stress, and the $\bar{B}$ here is the same as before.

2 Result

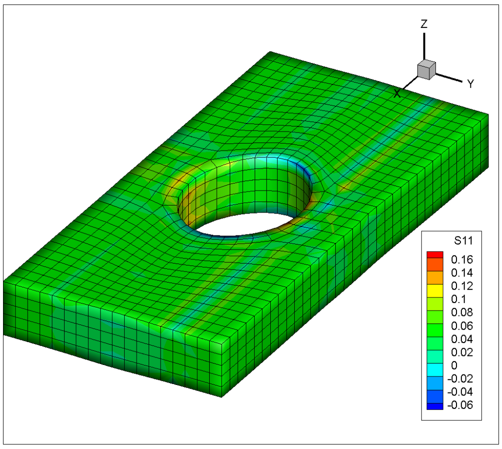
I used this UEL to solve hole-in-plate problem. To show the difference between normal material and incompressible material, I ran three simulations with three sets of moduli constants. For the material to be incompressible, I used $\mu = 0.1$ and $K = 5000$, which is equivalent to $\nu = 0.49999$. And for the material to be near-incompressible, I used $\mu = 1$ and $K = 100$, which is equivalent to $\nu = 0.495$ as comparison. Last, I used $\mu = 5$ and $K = 50$, which is equivalent to $\nu = 0.45$, to represent compressible material.

The figure below shows the comparison between two elements. First row is $\nu = 0.49999$, the middle row is $\nu = 0.495$, and the bottom row is $\nu = 0.45$.
From the top row figures, we can see that the displacement and stress field predicted by b-bar element and normal element is totally different. The stress field for normal element is almost uniform throughout the material, yet b-bar represents the right field. Also, the stress values for normal element failed too, predicting much larger values than b-bar. Last but not least, we can see that under the same loading, b-bar element predicts much less displacement than normal element, which is more reasonable since the Young's modulus is huge here.

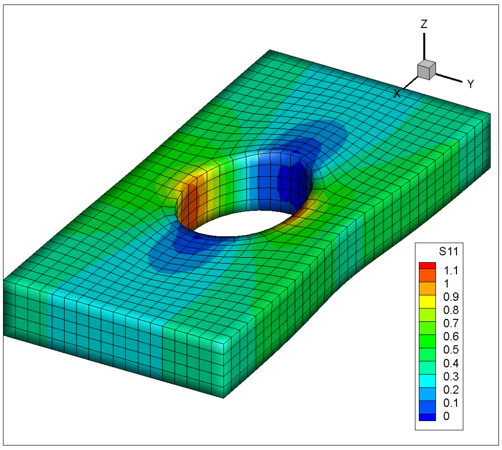
The middle row figures are for near-incompressible material. We can see that



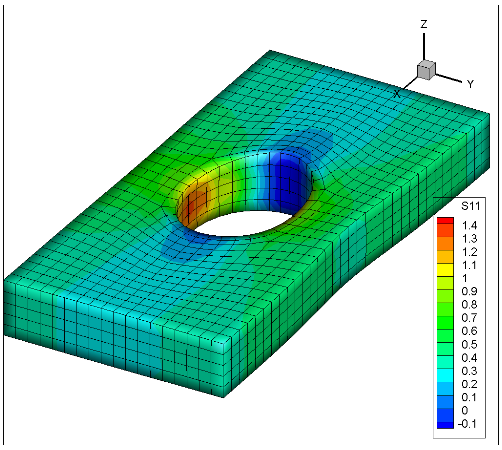
(a) B-bar element for incompressible



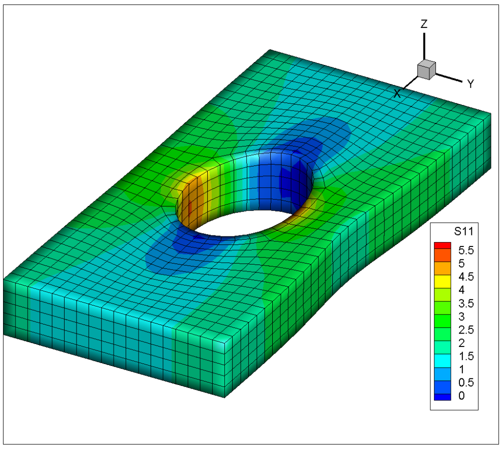
(b) Normal element for incompressible



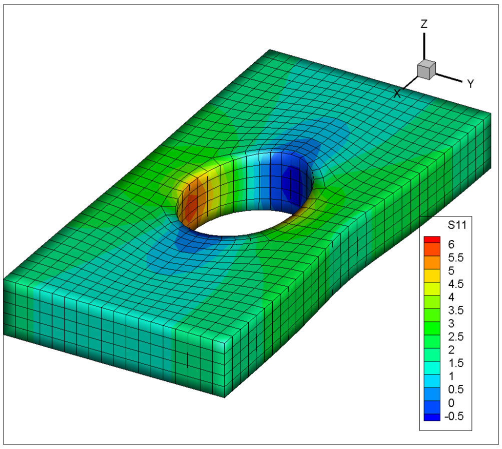
(c) B-bar element for near-incompressible



(d) Normal element for near-incompressible



(e) B-bar element for compressible



(f) Normal element for compressible

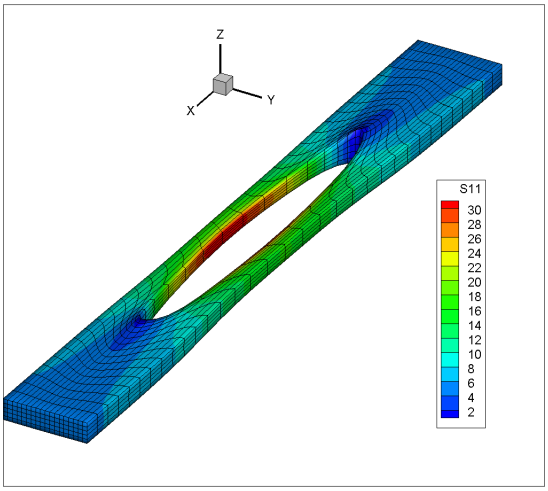
even for material that has a shear modulus close to 0.495, normal element can already give reasonable answers. The two figures are alike, and the predicted stress is off by a small amount. So we can say that normal element has a wide range of application, only in extreme cases will it be false.

The bottom row figures are almost the same and the stress are almost identical. This is for material with $\nu = 0.45$, which is totally compressible.Therefore for compressible materials the b-bar element does not differ from normal element and they can both give correct answer.

3 Conclusion and future work

For incompressible and near-incompressible materials, it is clear that b-bar element is better than normal material, since it avoids the effect of volumetric locking. But we can also see that normal element can give rather good result even for $\nu = 0.495$, which is very high in most cases, so it is also very applicable in simulation.

Due to limited time, the project only includes b-bar element as a fixation to volumetric locking. Future work could include the implementation of hybrid element and other method that solves this issue, and compare the result with b-bar element to show the difference.



Relevant code is on Github: https://github.com/Sijunniu/EN234_FEA