Final Project Report for ENGN2340

Sound Propagation in Porous Media

---Numerical simulation based on MATLAB

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EN-2340 Final Project

Sound Propagation in Porous Media

Abstract

Sound propagation in porous media is an important issue in engineering. In the current research, a MATLAB based simulation has been introduced to calculate the acoustic field in some structures. The first part of this article is to recall the governing equation of wave field and the corresponding variation problem. Then, a detailed description about the algorithm has been provided. As a test of the accuracy of the program, five different examples have been provided and compared with the corresponding analytical solutions.

I. Introduction

Noise pollution is a severe issue in the modern world. In order to decrease the influence of harmful sound wave on the environment, people usually use porous material as a kind of building material to absorb the noise. Porous media, included porous foam and sintered metal fiber material [1], is a type of functional material. Because of the micro pores inside the material, porous media can be used for energy absorption and sound absorption. Rayleigh [2] and Crandall [3] demonstrated that the micro channels can work efficiently in viscous dissipation and heat dissipation. Because of the high potential of the porous media in sound absorption, porous media is thoroughly studied in the previous decades[4, 5].

Porosity is an essential property of porous media. It is defined as the ratio of void pores and the total volume. In most cases, the porous media is considered as uniform. That means that it only has one porosity. However, more researches have been done on the double porosity material. It is proved that the double porosity material has high performance in sound absorption in low frequency range. Different kinds of double-porosity material were proposed and developed in the recent years. Theses researches are rigorous and inspiring. However, they only consider double-porosity. A nature extension is to study the acoustical porosity for multi-porosity material. By designing the distribution of micro pores inside the material, we can more effectively control the performance of porous media.

For porous media with single porosity, the Johnson-Champoux-Allard-Pride-Lafarge model [6-10] has considerable accuracy. For porous media with double porosity, the Olny-Boutin model [11, 12] can be used to predict the acoustic properties. However, for porous media with multi-porosity and random distribution, it is almost impossible for us to build an analytical model. The most efficient way for us to investigate the acoustic properties of multi-porosity material is to use numerical ways. However, ABAQUS and COMSOL have no module for man-controlled distributed-material. In the current study, a MATLAB based simulation has been provided to analyze the sound propagation in porous media.

II. Theoretical Modeling

2.1 Physical field

- Constitutive equation

Equivalent fluid model [6, 13] has been utilized to describe the homogenous properties of porous media. To simplify the problem, the following analysis is performed after Fourier transformation on the variable *t*. The corresponding Helmholtz [14] equation is given by

$$\nabla^2 p + k^2 p = 0 \tag{1}$$

where, p is the pressure, $k = \omega\left(\rho/K\right)^{1/2}$ is the wave number, ω is the angular frequency, ρ is the effective density and K is the effective modulus. According to JCAPL model [6], the effective density can be calculated by

$$\rho = \rho_0\left[\alpha_\infty + \frac{v\phi}{j\omega q_0}G\left(\omega\right)\right], G\left(\omega\right) = \left[1 + \left(\frac{2\alpha_\infty q_0}{\phi\Lambda}\right)^2 \frac{j\omega}{v}\right]^{1/2} \tag{2}$$

where ρ_0 is the density of the fluid, α_∞ is the tortuosity, v is the kinematic viscosity, ϕ is the porosity, q_0 is the static viscous permeability, Λ is the characteristic viscous length. The effective modulus is given as

$$K = \gamma P_0 \Bigg/ \left[\gamma - \frac{\gamma - 1}{1 + \dfrac{v'\phi}{j\omega q'_0 G'\left(\omega\right)}}\right], G' = \left[1 + \left(\frac{2q'_o}{\phi\Lambda'}\right)^2 \frac{j\omega}{v'}\right]^{1/2} \tag{3}$$

where γ is the heat ratio, P_0 is the atmosphere pressure, v', q'_0, Λ' are the corresponding parameters in the temperature field.

- Boundary condition

There are two kinds of boundary condition considered in the current simulation:

(1) Dirichlet boundary condition

$$p = p_{\text{boundary}}, \text{ at } \partial\Omega_d \tag{4}$$

(2) Natural boundary condition

$$\left(\nabla \cdot \mathbf{n}\right) p_t = \vartheta, \text{ at } \partial\Omega_n \tag{5}$$

2.2 Variation problem (Mathematical background)

For Helmholtz equation [15, 16],

$$
\begin{aligned}
&\nabla^2\varphi + k^2\varphi = 0\\
&\varphi = \varphi_{\text{boundary}}, \ \ \text{at } \ \partial\Omega_d \\
&(\nabla\cdot\mathbf{n})\varphi = \vartheta, \ \ \text{at } \ \partial\Omega_n
\end{aligned} \tag{6}
$$

the corresponding variation problem is.

$$
I(\varphi) = \frac{1}{2}\iint\left[\left|\nabla\varphi\right|^2 - k^2\varphi^2\right]d\Omega + \int_{\partial\Omega}\vartheta\varphi d\Gamma \tag{7}
$$

In the current problem, we only consider the solid hard wall $\vartheta = 0$, so the second term $\int_{\partial\Omega}\vartheta\varphi d\Gamma$ can be neglected. By using the triangular element, we shall have the following shape function

$$
N^a(x_1,x_2) = \frac{(x_2 - x_2^b)(x_1^c - x_1^b) - (x_1 - x_1^b)(x_2^c - x_2^b)}{(x_2^a - x_2^b)(x_1^c - x_1^b) - (x_1^a - x_1^b)(x_2^c - x_2^b)} \tag{8}
$$

$$
N^b(x_1,x_2) = \frac{(x_2 - x_2^c)(x_1^a - x_1^c) - (x_1 - x_1^c)(x_2^a - x_2^c)}{(x_2^b - x_2^c)(x_1^a - x_1^c) - (x_1^b - x_1^c)(x_2^a - x_2^c)} \tag{9}
$$

$$
N^c(x_1,x_2) = \frac{(x_2 - x_2^a)(x_1^b - x_1^a) - (x_1 - x_1^a)(x_2^b - x_2^a)}{(x_2^c - x_2^a)(x_1^b - x_1^a) - (x_1^c - x_1^a)(x_2^b - x_2^a)} \tag{10}
$$

The subscript (*a*,*b*,*c*) refers to the three nodes of the elements (Distinguish from the subscript (1,2,3) which refers to the three integration points). Then, φ_e in the element can be written in the form

$$
\varphi_e(x_1,x_2) = N^a(x_1,x_2)\varphi^a + N^b(x_1,x_2)\varphi^b + N^c(x_1,x_2)\varphi^c \tag{11}
$$

For a single element, the function $I(\varphi_e)$ can be written as

$$
I(\varphi_e) = \left\{\frac{1}{2}\sum_{i=a}^{c}\sum_{j=a}^{c}\varphi^{(i)}\varphi^{(j)}\iint\nabla N_i\nabla N_j dS\right\} - \left\{\frac{k^2}{2}\sum_{i=a}^{c}\sum_{j=a}^{c}\varphi^{(i)}\varphi^{(j)}\iint N_i N_j dS\right\} \tag{12}
$$

We can define $C_{ij} = \iint\nabla N_i\nabla N_j dS$ and $T_{ij} = \iint N_i N_j dS$, then

$$
I(\varphi_e) = \frac{1}{2}\varphi_e^T C_e\varphi_e - \frac{k^2}{2}\varphi_e^T T_e\varphi_e \tag{13}
$$

Assemble the whole elements, we can have

$$
I(\varphi) = \frac{1}{2}\varphi^T C\varphi - \frac{k^2}{2}\varphi^T T\varphi \tag{14}
$$

To minimize the function $I(\varphi)$, we shall have

$$\frac{1}{2}\left(C-k^2T\right)\varphi = K\varphi = 0 \tag{15}$$

In local coordinate, the three nodes for the elements are $(\xi_1,\xi_2)^a = (1,0)$, $(\xi_1,\xi_2)^b = (0,1)$ and $(\xi_1,\xi_2)^c = (0,0)$. Therefore, the three shape functions for local coordinate are

$$N^a = \xi_1, N^b = \xi_2, N^c = 1-\xi_1-\xi_2 \tag{16}$$

The three integration points are set as

$$\begin{aligned}
&\xi_1^1 = 0.5, \xi_2^1 = 0, w_1 = 1/6\\
&\xi_1^2 = 0, \xi_2^2 = 0.5, w_2 = 1/6\\
&\xi_1^3 = 0.5; \xi_2^3 = 0.5, w_3 = 1/6
\end{aligned} \tag{17}$$

For a single element,

$$\varphi_e = \begin{bmatrix} \varphi^a & \varphi^b & \varphi^c \end{bmatrix}^T \tag{18}$$

the *C* matrix can be determined by

$$C_{ij} = \sum_{k=1}^{3} \nabla N_i^k \nabla N_j^k w_k \eta_k \tag{19}$$

where w_k is the weight for the integration point k, $\eta_k = \det\left(\partial x/\partial \xi\right)$ is the term that measure the area of the element. For a single integration points, the $\tilde{B}_k$ matrix for the integration point *k* is

$$\tilde{B}_k = \begin{bmatrix} \partial N^a/\partial x_1 & \partial N^a/\partial x_2 \\ \partial N^b/\partial x_1 & \partial N^b/\partial x_2 \\ \partial N^c/\partial x_1 & \partial N^c/\partial x_2 \end{bmatrix} \tag{20}$$

Then *C* matrix can be expressed as

$$C = \sum_{k=1}^{3} \tilde{B}_k \tilde{B}_k^T w_k \eta_k \tag{21}$$

For a single element, the T matrix is

$$T_{ij} = \int\int N_i N_j dS = \sum_{i=1}^{3} N_i^k N_j^k w_k \eta_k \tag{22}$$

Take,

$$\widehat{B}_k = \begin{bmatrix} N^a \\ N^b \\ N^c \end{bmatrix} \tag{23}$$

Therefore

$$T = \sum_{k=1}^{3} \widehat{B}_k \widehat{B}_k^T w_k \eta_k \tag{24}$$

Assemble the two terms, we shall have

$$K = \frac{1}{2} \sum_{k=1}^{3} \left(\tilde{B}_k \tilde{B}_k^T - k^2 \widehat{B}_k \widehat{B}_k^T \right) w_k \eta_k \tag{25}$$

For the Dirichlet boundary condition $\varphi = \varphi_{\text{boundary}}$, we can just make the corresponding row in the stiffness matrix to be zero. And set the corresponding value of the residual equal to $\varphi_{\text{boundary}}$.

For the Natural boundary condition $(\nabla \cdot \mathbf{n})\varphi = 0$, we can just leave it free. To obtain the aim function φ, we only need to solve

$$\varphi = r / K \tag{26}$$

- Mesh:

The current simulation is built based on a simple 2D rectangle field, as below

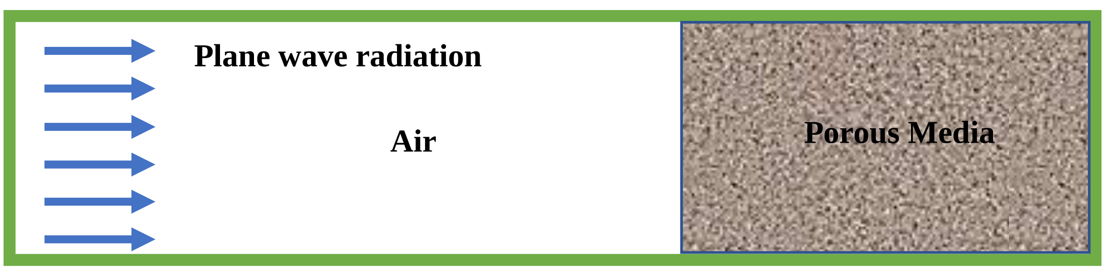

Figure 1 Sound propagation in a standing wave tube

- Element:

We shall use the basic 2-D triangular element: three nodes, three integration points.

- Boundary condition:

Dirichlet boundary condition on the left and solid wall on the other boundary.

- Time step:

This is a frequency-based simulation.

- Iteration:

$\varphi = r / K$ is solved directly.

III. Result and Discussion

Several models have been provided to validate the current frequency-based simulation.

3.1 Example One: Sound propagation in a micro channel

3.1.1 Problem Description:

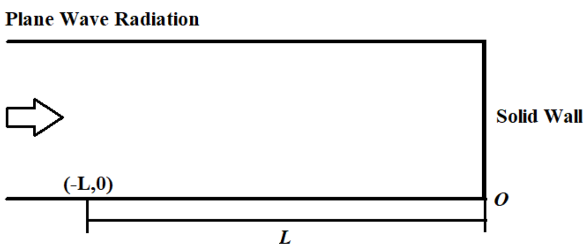

Figure 2 Plane wave radiation in a micro channel (Example 1)

A plane sound propagates from left to right. At the position (-L,0), the pressure is 1pa. At the position (0,0), it is solid wall.

3.1.2 Theoretical Modeling:

It is a 1-D problem. According to the theory of Pressure-acoustics, the pressure distribution can be calculated by

$$p = A\exp\left[j\left(-kx+\omega t\right)\right] + A\exp\left[j\left(kx+\omega t\right)\right] \tag{27}$$

where, $A\exp\left[j\left(-kx+\omega t\right)\right]$ is the wave propagating in x direction, $A\exp\left[j\left(kx+\omega t\right)\right]$ is the reflection wave propagating in -x direction. The origin of the coordinate is set on the right side. Therefore, at the point (x=0), the pressure is

$$p = 2A \tag{28}$$

At the point (x=-L), the pressure is

$$p = 2A\cos kL = 1 \tag{29}$$

Therefore amplitude for the input wave can be calculated by,

$$A = \frac{1}{2\cos kL} \tag{30}$$

The distribution of pressure is

$$p = \frac{\cos kx}{\cos kL} \tag{31}$$

3.1.3 Results:

In the simulation, we set $\text{Length} = 0.05 \text{ m}, \text{Height} = 0.001 \text{ m}, \text{Frequency} = 1000 \text{ Hz}$. The following results are all plotted by MATLAB:

The mesh of the structure

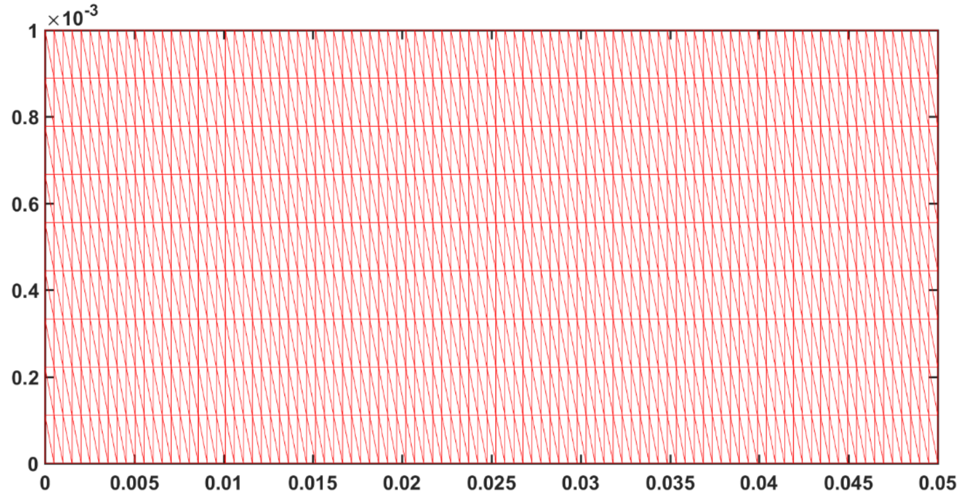

Figure 3 Triangular mesh of the micro channel (Example 1)

The contour of pressure

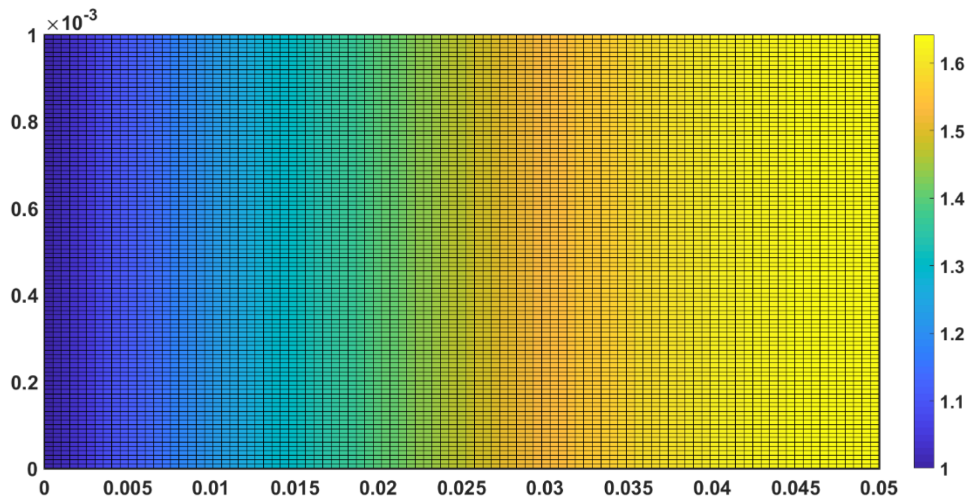

Figure 4 Contour of the pressure field (Example 1)

The pressure field as the function of x.

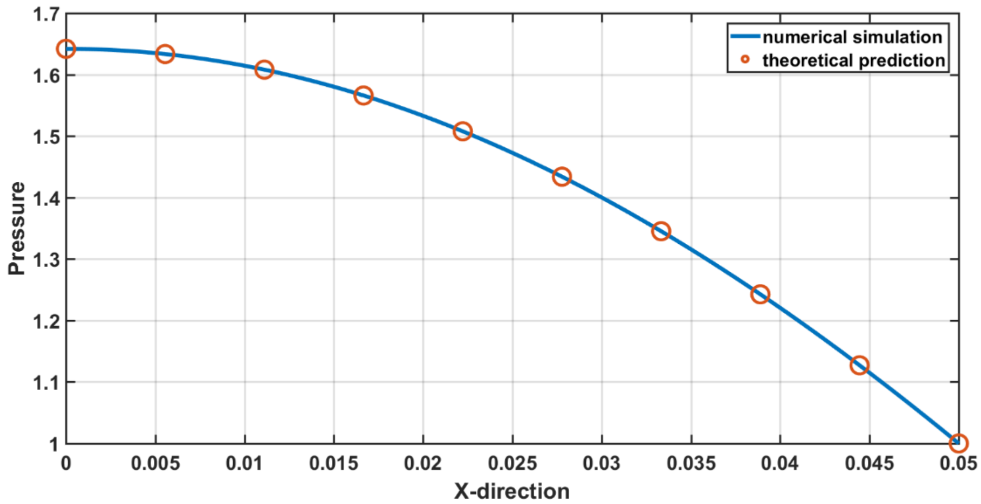

Figure 5 Comparisons of the numerical simulation and the theoretical prediction of the pressure as a function of x. (Example 1)

The above figures demonstrated that the numerical solution agree well with the theoretical prediction.

3.2 Example Two: Sound wave passing a cylinder

3.2.1 Problem description:

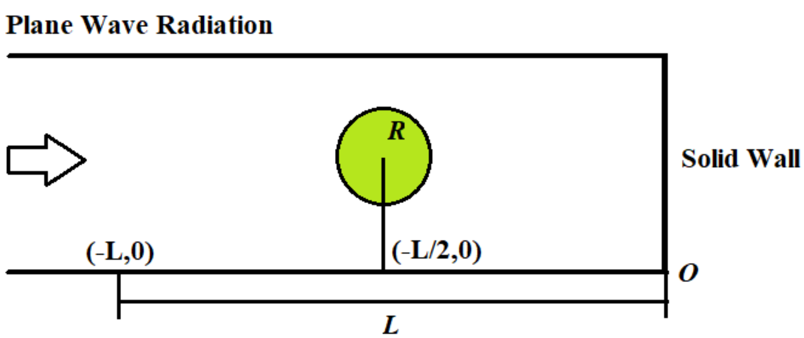

Figure 6 Sound wave passing a cylinder (Example 2)

A plane sound propagates from left to right. At the position (-L,0), the absolute pressure is 1pa. At the position (0,0), there is sound hard wall. At the middle of the structure, there is a solid cylinder. Try to get the distribution of pressure in the system.

3.2.2 Results

In the simulation, we set $\text{Length} = 0.1\ \text{m}$, $\text{Height} = 0.05\ \text{m}$, $\text{Radius} = 0.0125\ \text{m}$ and $\text{Frequency} = 10^2, 10^4\ \text{Hz}$. The following results are all plotted by MATLAB:

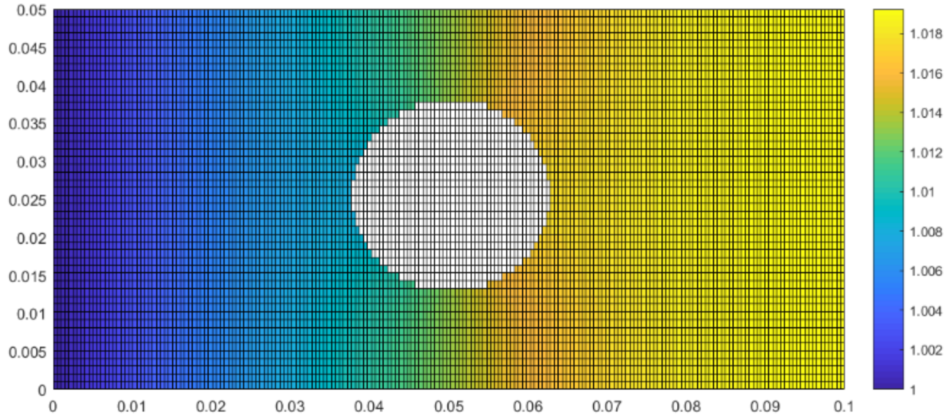

Figure 7 Contour of the pressure field, low frequency (Example 2)

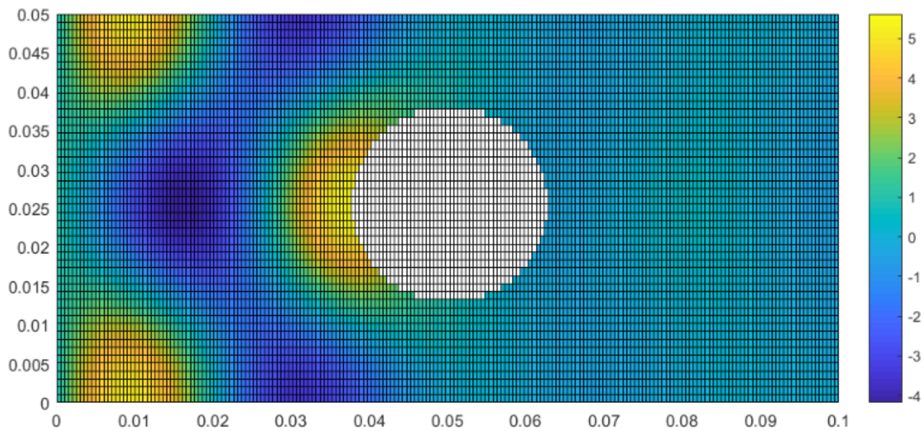

Figure 8 Contour of the pressure field, high frequency (Example 2)

Figure 7 shows that, it is much easier for low-frequency sound wave to pass the cylinder, because the wavelength of the sound is relatively larger than the characteristic length of the cylinder. In this case, there will be more diffraction. Figure 8 shows that when the wavelength of the sound is of the same order as the cylinder, there will be more reflection.

3.3 Example Three: Sound propagation in a porous media

3.3.1 Problem description

The structure parameters are the same as example 1. For convenience, the effective density and effective modulus are given as below (i.e. they the build based on micro-slits model):

$$\rho = \rho_0 \left[1 - \frac{\tanh\left(s' j^{1/2} \right)}{s' j^{1/2}} \right]^{-1}, s' = \left(\frac{\omega \rho_0 R^2}{\eta} \right)^{1/2} \tag{32}$$

$$K=\gamma P_0\left[1+(\gamma-1)\frac{\tanh\left(Bs' j^{1/2}\right)}{Bs' j^{1/2}}\right]^{-1}, s'=\left(\frac{\omega\rho_0 R^2}{\eta}\right)^{1/2} \tag{33}$$

where, η is the dynamic viscosity of the fluid, R is the radius of the pore, $\gamma=1.4$, P_0 is the atmosphere pressure, j is the imaginary unit, $B=\sqrt{\mathrm{Pr}}$ is the square root of Prantle number (0.71). In the current research, we use $a=0.0005\ \mathrm{m}$.

3.3.2 Theoretical modeling

According to Eq. (31), the distribution of pressure is,

$$p=\frac{\cos kx}{\cos kL} \tag{34}$$

It should be noticed that k is a complex number, so the pressure here is a complex number, it not only reflects the amplitude, but also the phase. Therefore, we shall plot real term and imaginary term separately.

3.3.3 Results

In the simulation, we will set $\text{Length}=0.1\ \mathrm{m}$, $\text{Height}=0.05\ \mathrm{m}$ and $\text{Frequency}=1000\ \mathrm{Hz}$. The following results are all plotted by MATLAB:

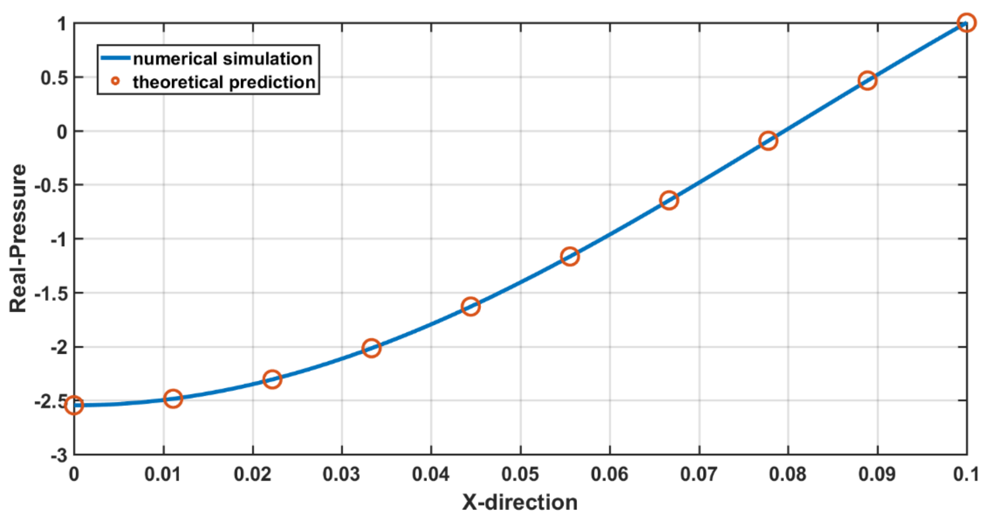

Figure 9 The real part of the pressure as a function of x (Example 3)

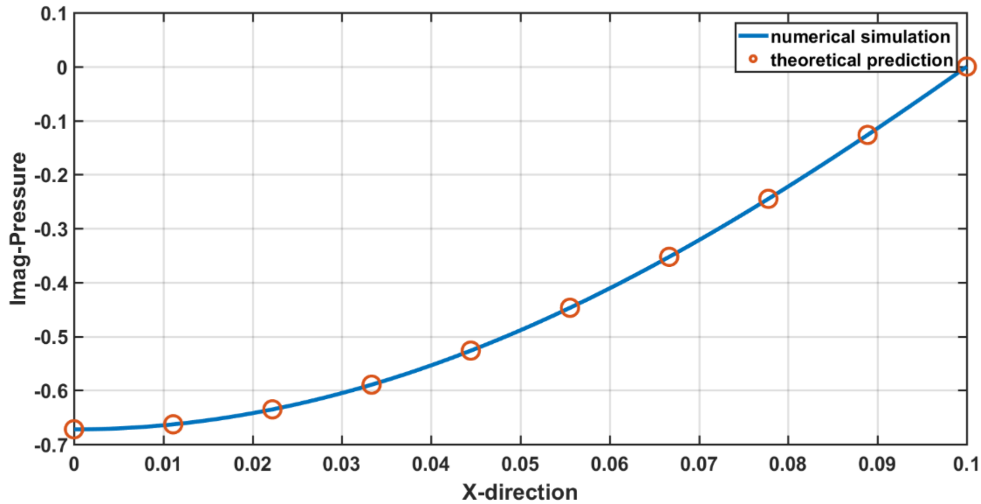

Figure 10 The imaginary part of the pressure as a function of x (Example 3)

3.4 Example Four: Sound Propagation in material with double porosity. Low contrast and High contrast.

3.4.1 Problem description

The sound propagation in double porosity material is quite different from the single porous media. When the characteristic lengths of the two materials are incomparable, there will be pressure diffusion effect. This is called high contrast model (note: there is a characteristic frequency). When the characteristic lengths of the two materials are comparable, the pressure diffusion effect is weak enough to be neglected. This is called low contrast model. In the following simulations, we will simulate the two cases separately.

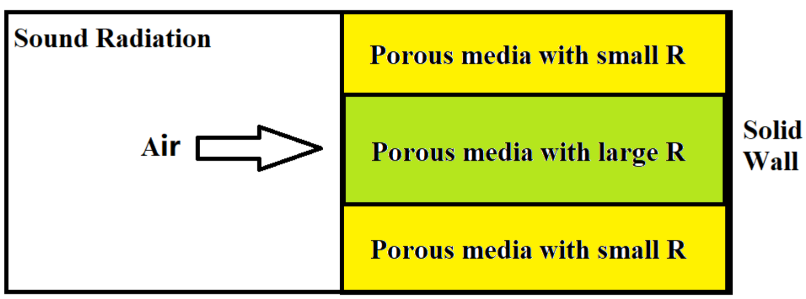

Figure 11 Sound propagation in double porosity porous media (Example 4)

I have set a background ground pressure field in the front of the porous media. This can be approximated as the plane wave radiation.

3.4.2 Results

- Low contrast:

In the simulation, there are 50×50 nodes in the mesh. The structural parameters are given by $\text{Length} = 0.02\text{ m}$, $\text{Width} = 0.02\text{m}$, $\text{R\_1} = 0.005\text{ m}$, $\text{R\_2} = 0.01\text{ m}$ $\text{Frequency} = 3550\text{Hz}$. The following results are all plotted by MATLAB:

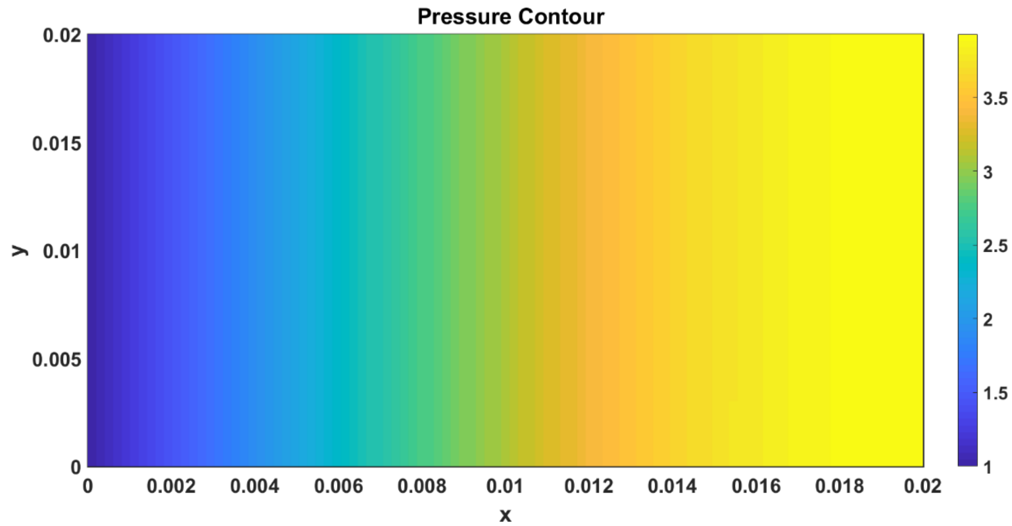

Figure 12 Pressure contour in low contrast (Example 4)

Figure 12 shows the pressure distribution is uniform. There is no pressure diffusion effect.

- High contrast (Pressure diffusion effect)

In the simulation, there are 50×50 nodes in the mesh. The structural parameters are $\text{Length} = 0.02\text{ m}$, $\text{Width} = 0.02\text{m}$, $\text{R\_1} = 0.00002\text{ m}$, $\text{R\_2} = 0.01\text{ m}$ $\text{Frequency} = 3550\text{Hz}$. The following results are all plotted by MATLAB:

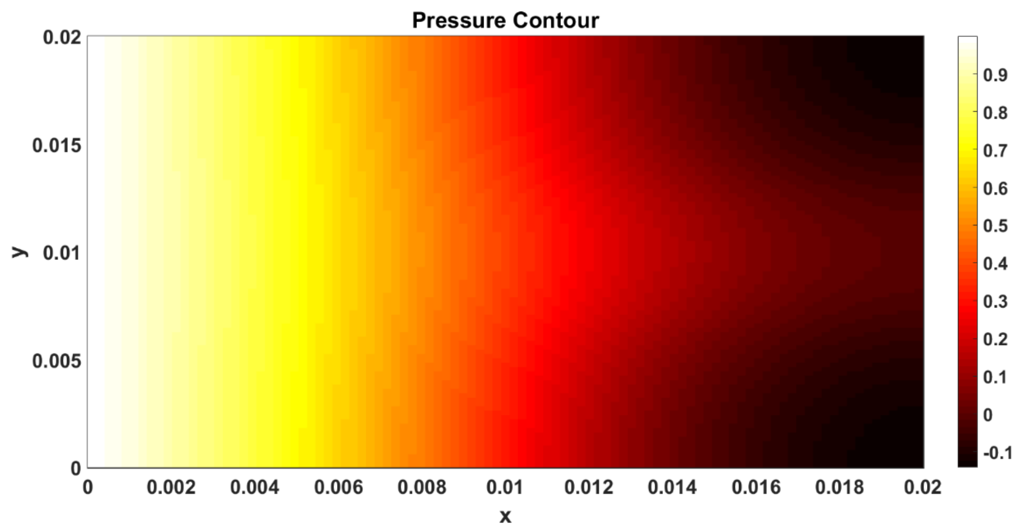

Figure 13 Pressure contour in high contrast (Example 4)

Figure 13 shows that the pressure distribution is not uniform. The pressure diffusion effect is significant at the boundary between two porous media.

3.5 Example Five: Sound Propagation in porous material with multi-porosity

3.5.1 Problem description

The porous materials with multi porosity have multi sizes of pores. In the current problem, we consider the following distribution of characteristic radius (*R*) of the porous media:

$$R(x,y) = \left(0.00001 + 0.001 \cdot \frac{x}{\text{Length}} \cdot \frac{y}{\text{Height}}\right) \text{ m}$$

where length is the length of the porous media, height is the height of the porous media. The origin is put at the lower left side of the porous media. As introduced in Figure 11, there is also a background pressure field in the front of the porous material.

3.5.2 Results

In the simulation, there are 50×50 nodes in the mesh. The structural parameters are $\text{Length} = 0.02\text{ m}, \text{Height} = 0.02\text{m Frequency} = 3550\text{Hz}$. The following results are all plotted by MATLAB:

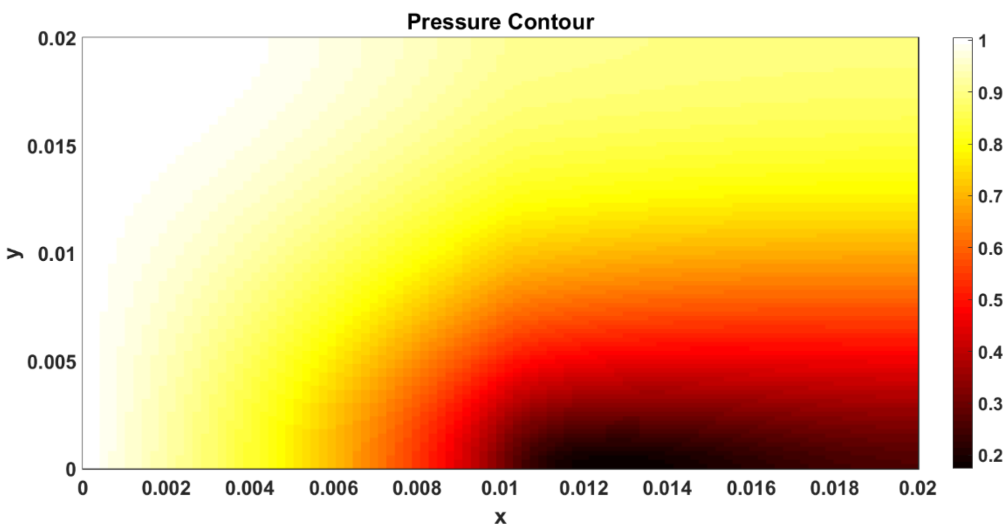

Figure 14 Pressure contour of multi-porosity material (Example 5)

Figure 14 shows that it is much easier for sound wave passing porous media with larger characteristic size. This is because viscos resistivity is small when the radius of the pore is large.

IV. Conclusion

In the current project, I have done the following work:

(1) Study the Variation problem of the Helmholtz equation with two different kinds of boundary condition;
(2) Build the corresponding finite element model and implement it in MATLAB;
(3) Test the program in five different cases: wave propagation in a rectangular channel; sound wave passing a cylinder; wave propagation in material with single porosity, double porosity and multi porosity.

There are also some problems unsolved, such as how to add acoustic source to the system or how to add viscos dissipation and thermal dissipation to the problem, etc. All of these challenges need deeper understanding of the acoustic field.

V. Acknowledgement

I greatly appreciated Prof. Bower's instruction on this course. From the lecture notes and homework assignment, I have learnt a lot about programing styles and data structures. The course EN2340 is somewhat difficult, but it really provides me a lot of useful knowledge. In addition, I shall thank my classmates Sijun, Harkirat, etc. Discussion with them is intriguing and inspiring.

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