Division of Engineering
Brown University

EN40: Dynamics and Vibrations

Midterm Examination
Tuesday March 9 2010

NAME: IMA Genius ______________________________

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

IMAG ____________

1 (4 points) ____________

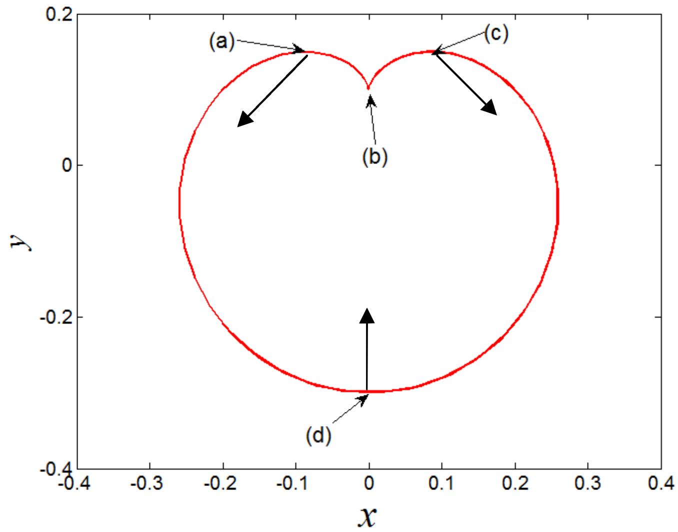
2 (4 points) ____________

3. (10 points) ____________

4. (12 points) ____________

5. (10 points) ____________

TOTAL (40 points) ____________



1. The figure shows the trajectory of a particle in a Penning trap, for a particular choice of the electric and magnetic fields that trap the particle. The particle remains in the (x,y) plane at all times, and move from a to b to c to d…

At point (a), the particle's speed is decreasing
At point (c), the particle's speed is increasing
At point (d), the particle's speed is a maximum

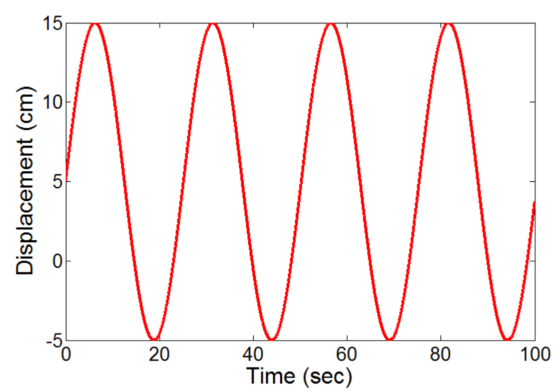
1.1 Draw arrows on the figure at points (a), (c), and (d) to show the approximate direction of the particle's acceleration vector.

[1 POINT EACH]

1.2 What is the particle's speed at point (b)? The trajectory is vertical, so the particle must have zero vertical velocity. The direction of motion of the particle must reverse, so at (b) the vertical component of velocity must also be zero. So the total speed is zero.

[1 POINT]

2. The figure shows a vibration measurement from a displacement transducer. The vibration may be assumed to be harmonic. Estimate



(a) The period of oscillation

There are 4 cycles in 100 sec, so period is 100/4=25 sec

(b) The angular frequency of oscillation

Angular frequency is related to period by

$$\omega = 2\pi / T = 2\pi / 25 \quad rad / \sec$$

(c) The amplitude of the velocity

The amplitude of the displacement is 10 cm. The velocity amplitude is related to displacement amplitude by $\Delta V = \omega \Delta X = 20\pi / 25 \quad cm / \sec$

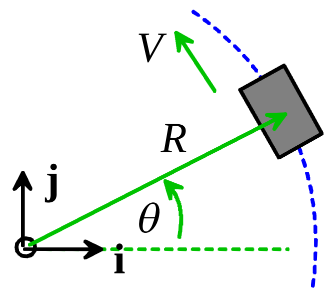
(d) The amplitude of the acceleration

The acceleration amplitude is related to velocity amplitude by $\Delta A = \omega \Delta V = 10.(2\pi/25)^2 \quad cm/\sec^2$

[1 POINT EACH]

3. An airport 'people mover' travels at constant speed *V* around a circular path with radius *R*.

3.1 Write down the position vector of the vehicle in terms of *R* and the angle θ shown in the figure.



Trig gives $\mathbf{r} = R\cos\theta\mathbf{i} + R\sin\theta\mathbf{j}$

[1 POINT]

3.2 Hence, calculate formulae for the velocity and acceleration vectors for the vehicle, in terms of *R*, *V*, and θ, expressing your answer as components in the basis shown.

Note that θ varies with time. Differentiate the position vector, using the Chain rule

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -R\sin\theta\mathbf{i}\frac{d\theta}{dt} + R\cos\theta\frac{d\theta}{dt}\mathbf{j} = R\frac{d\theta}{dt}\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right).$$

Note that $\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)$ is a unit vector, so $Rd\theta/dt$ is the magnitude of the velocity (i.e. the speed *V*) and therefore $R\frac{d\theta}{dt} = V$. Thus

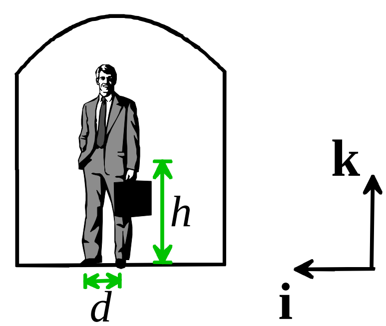
$$\mathbf{v} = V\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)$$

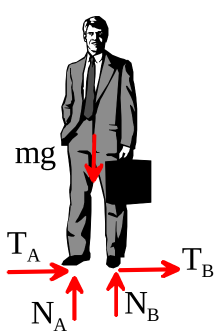
Differentiate again to find the acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = V\left(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right)\frac{d\theta}{dt} = \frac{V^2}{R}\left(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}\right)$$

[2 POINTS]

3.4 The figure shows a passenger inside the car, at the instant when $\theta = 0$. His center of mass is a height *h* above the floor, and he stands with feet a distance *d* apart, facing in the direction of motion of the vehicle. There is sufficient friction between the floor and his feet to prevent slip. Draw the forces acting on the passenger on the figure provided below





[2 POINTS]

3.5 By considering the motion of the passenger at the instant when $\theta = 0$, determine formulae for the reaction forces exerted on the passenger by the floor of the vehicle, in terms of *m*, *g*, *V* , *R*, *d* and *h*. Not all the forces can be determined uniquely.

Substituting $\theta = 0$ into the acceleration formula from 3.2, and writing down Newton's law gives

$$\mathbf{F} = -(T_A + T_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{k} = m\mathbf{a} = -m\frac{V^2}{R}\mathbf{i}$$

We can get another equation of motion by idealizing the passenger as a massless frame, in which case moments about the COM must vanish, i.e.

$$\left(T_A + T_B\right)h\mathbf{j} + (N_B - N_A)\frac{d}{2}\mathbf{j} = \mathbf{0}$$

The **i** component of the first equation gives.

$$\left(T_A + T_B\right) = \frac{mV^2}{R}$$

Using the **k** component of the first equation, and substituting for $\left(T_A + T_B\right)$ in the second gives

$$(N_A + N_B - mg) = 0$$

$$\frac{mV^2}{R}h + (N_B - N_A)\frac{d}{2} = 0$$

These can be easily solved to give

$$N_A = \frac{mg}{2} + \frac{mV^2}{R}\frac{h}{d} \qquad N_B = \frac{mg}{2} - \frac{mV^2}{R}\frac{h}{d}$$

[3 POINTS]

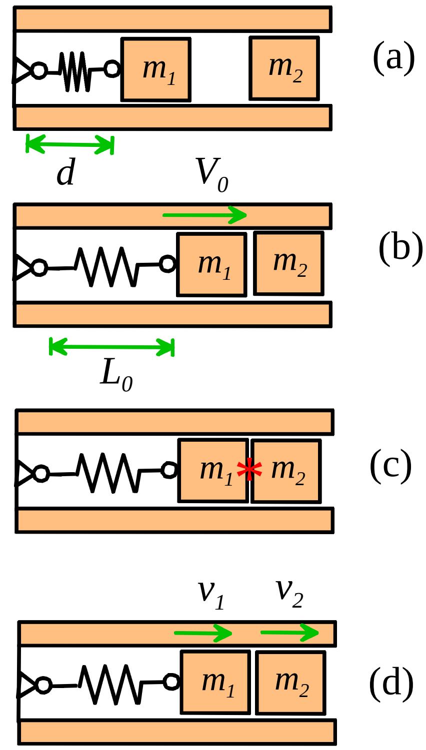
3.6 Finally, calculate an expression for the minimum allowable radius of the path for the passenger to remain standing, in terms of *V*, *g*, *h* and *d*.

The passenger tips over if his feet lose contact with the ground. Contact is lost if the reaction force is zero or negative. The preceding part of the problem shows that N_A is always positive, but N_B will be zero if the radius is too small. So

$$N_B = \frac{mg}{2} - \frac{mV^2}{R}\frac{h}{d} > 0 \Rightarrow R > \frac{2V^2}{g}\frac{h}{d}$$

[2 POINTS]

4. The figure shows a proposed design for a spring-loaded catapult. It operates as follows: (a) The spring is compressed to a length d and then released from rest; (b) the spring returns to its unstretched length, accelerating mass m_1 to a speed V_0; (c) immediately after this point masses m_1 and m_2 collide; (d) causing mass m_2 to be expelled from the muzzle with speed v_2



The spring has stiffness k and un-stretched length L_0, and the collision between the two masses can be characterized by a restitution coefficient e.

4.1 Write down the potential energy of the system in state (a).

The PE is just the energy of the spring, i.e.

$$V = \frac{1}{2}k(L_0 - d)^2$$

[1 POINT]

4.2 Hence, calculate a formula for the speed of mass m_1 just before impact (b), in terms of k, L_0, m_1 and d.

This is a conservative system, so PE+KE is constant. At the instant just before impact, the spring returns to its unstretched length, and the KE is

$$T = \frac{1}{2}m_1 v_1^2$$

Thus

$$\frac{1}{2}k(L_0 - d)^2 = \frac{1}{2}m_1 v_1^2 \Rightarrow v_1 = \sqrt{\frac{k}{m_1}}(L_0 - d)$$

[2 POINTS]

4.3 Deduce expressions for the speeds v_1 and v_2 of the two masses just after the collision (d), in terms of k, L_0, m_1, m_2, e and d.

Momentum is conserved during the impact, so $m_1\sqrt{\frac{k}{m_1}}(L_0 - d) = m_1 v_1 + m_2 v_2$

In addition, velocities before and after impact are related by the restitution coefficient

$$e\sqrt{\frac{k}{m_1}}(L_0 - d) = (v_2 - v_1)$$

These two equations can be solved for the two velocities, with the result

$$v_2 = \frac{(1+e)m_1}{m_1+m_2}\sqrt{\frac{k}{m_1}}(L_0-d) \qquad v_1 = \frac{(m_1-em_2)}{m_1+m_2}\sqrt{\frac{k}{m_1}}(L_0-d)$$

[4 POINTS]

4.4 Show that the speed of mass m_2 is optimized if $m_1 = m_2$

From the preceding part,

$$v_2 = \frac{\sqrt{m_1}}{m_1+m_2}(1+e)\sqrt{k}(L_0-d)$$

We want to find the value of m_1 that maximizes this (for a fixed projectile mass m_2). Differentiate with respect to m_1 and set the derivative to zero

$$\frac{dv_2}{dm_1} = 0 \Rightarrow \frac{1}{2\sqrt{m_1}\left(m_1+m_2\right)} - \frac{\sqrt{m_1}}{\left(m_1+m_2\right)^2} = 0$$

$$\Rightarrow \frac{m_1+m_2-2m_1}{2\sqrt{m_1}\left(m_1+m_2\right)^2} = 0 \Rightarrow m_1 = m_2$$

[**2 POINTS]**

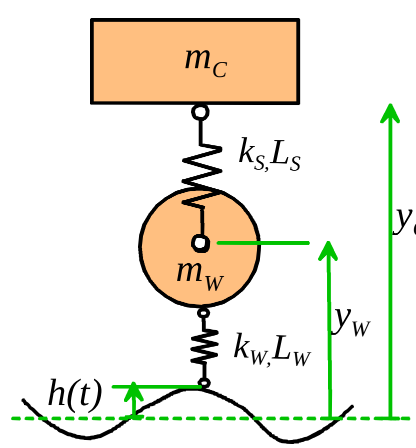
4.5 Finally, compute a formula for the energy efficiency of the optimal design.

The energy efficiency is the ratio of the KE of the projectile to the initial PE in the spring. With $m_1 = m_2$ this is

$$\frac{\frac{1}{2}m_2v_2^2}{\frac{1}{2}k(L_0-d)^2} = \frac{m_2m_1}{\left(m_1+m_2\right)^2}(1+e)^2 = \frac{(1+e)^2}{4}$$

[3 POINTS]

5. The figure shows an idealization of a vehicle's suspension system. Mass m_c represents the body of the vehicle, while mass m_w represents the wheel. The spring with stiffness k_s and unstretched length L_s represents the shock absorbers, while the spring with stiffness k_w and unstretched length L_w accounts for the deformation of the car tire. As the car drives over a rough road, the base of this spring vibrates vertically with a time dependent displacement *h(t)*. The motion of the system will be described by the height y_w and y_c of the wheel and car, respectively.



5.1 Write down an expression for the acceleration of the two masses, in terms of time derivatives of the heights y_w and y_c. Both masses may be assumed to have a constant horizontal velocity. You don't need to use

Newton’s laws to answer this part.

The vertical component of acceleration is simply
$$a_w = \frac{d^2 y_w}{dt^2}$$
$$a_c = \frac{d^2 y_c}{dt^2}$$

[2 POINTS]

5.2 Draw the forces acting on the two masses on the figure provided.

[3 POINTS]

5.3 Hence, derive equations of motion for y_w and y_c. Both masses can be idealized as point masses.

Vertical components of F=ma for the two masses gives

$$m_c a_c = m \frac{d^2 y_c}{dt^2} = -m_c g - F_s$$
$$m_w a_w = \frac{d^2 y_w}{dt^2} = F_s - m_{wg} - F_w$$

And the spring force law is $\begin{matrix} F_s = k_s(y_c - y_w - L_s) \\ F_w = k_w(y_w - h - L_w) \end{matrix}$. Substituting this into the formula and rearranging the result gives

$$\frac{d^2 y_c}{dt^2} = -g - \frac{k_s}{m_c}(y_c - y_w - L_s)$$
$$\frac{d^2 y_w}{dt^2} = \frac{k_s}{m_w}(y_c - y_w - L_s) - g - \frac{k_w}{m_w}(y_w - h - L_w)$$

[3 POINTS]

5.4 Arrange the equations of motion into a form that could be integrated numerically using the MATLAB ODE solver.

For a MATLAB solution, we would have to turn the second derivatives into first derivatives by introducing the velocities as additional unknowns. This gives

$$\frac{d}{dt}\begin{bmatrix} y_c \\ y_w \\ v_c \\ v_w \end{bmatrix} = \begin{bmatrix} v_c \\ v_w \\ -g - k_s(y_c - y_w - L_s)/m_c \\ k_s(y_c - y_w - L_s)/m_w - g - k_w(y_w - h - L_w)/m_w \end{bmatrix}$$

[2 POINTS]