School of Engineering
Brown University

EN40: Dynamics and Vibrations

Final Examination
Tuesday May 15, 2011

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may use 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- *If you find you are unable to complete part of a question, proceed to the next part.*

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

____

1-10: (20 PTS) _____________

11: (10 PTS) _____________

12: (10 PTS) _____________

13: (10 PTS) _____________

14: (10 PTS) _____________

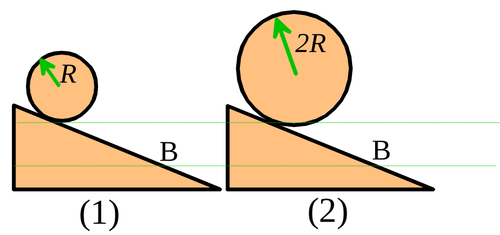
TOTAL (60 PTS) ___________

FOR PROBLEMS 1-10 WRITE YOUR ANSWER IN THE SPACE PROVIDED.

ONLY THE ANSWER APPEARING IN THE SPACE PROVIDED WILL BE GRADED. ILLEGIBLE ANSWERS WILL NOT RECEIVE CREDIT.

1. Two cylinders with equal mass density start at rest, and roll without slipping down an incline. Cylinder 1 has a radius *R* and cylinder 2 has a radius 2*R*. Which cylinder will have a higher velocity when it arrives at point B?

(a) Cylinder 1
(b) Cylinder 2
(c) Both the same
(d) This is one of life's unanswerable questions.



For a cylinder with general radius r the KE is $T = (I\omega^2 + mv^2)/2$ and since the cylinder rolls without sliding $\omega = v/r$. Energy conservation gives $mgh = (I/r^2 + m)v^2/2$ and finally recall $I = mr^2/2$ so that $gh = 3v^2/4 \Rightarrow v = 2\sqrt{gh/3}$. This is independent of the radius or mass of the cylinder. Or just remember Prof Franck's class demo!

Answer: (c)

2. The unit kg m^2/s is used for:

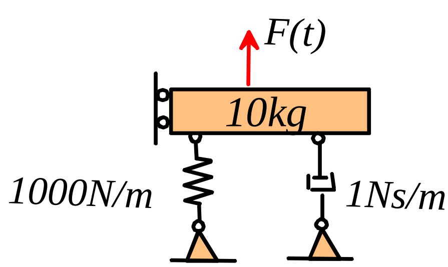
(a) Rotational Kinetic Energy
(b) Power
(c) Angular Momentum
(d) All of the above
(e) None of the above

The units are $(a) I\omega^2 = (kgm^2)/s^2$ (b) $P = Fv = (kgm/s^2)(m/s)$ $(c) h = rmv = m.kg.m/s$

Answer: (c)

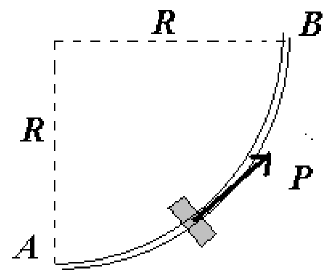
3. A vibration isolation table can be idealized as shown in the figure. It is subjected to a harmonic force *F(t)* with amplitude 0.1N and angular frequency 10 rad/s. The amplitude of vertical vibration is

(a) 0.1 mm
(b) 0.2 mm
(c) 1 mm
(d) 10 mm
(e) 20 mm
(f) None of the above



The natural frequency is $\omega = \sqrt{k/m} = 10 rad/s$. The system is therefore at resonance. The damping coefficient is $\zeta = 1/(2\sqrt{km}) = 1/200$. At resonance the amplitude is $(F_0/k)/(2\zeta) = 0.1 \times 200/1000/2 = 0.01m = 10mm$. **Answer: (d)**

4. A bead of mass m slides on frictionless ring of radius R in a vertical plane. The block is subjected to a vertical gravitational force mg as well as a force P=2mg that is always oriented along the direction of sliding. The block starts from rest at point A. What is the velocity of the block when it reaches point B?



(a) $v = \sqrt{2Rg}$

(b) $v = \sqrt{2(\pi - 1)Rg}$

(c) $v = \sqrt{4Rg}$

(d) $v = \sqrt{2\left(2\sqrt{2} - 1\right)Rg}$

(e) None of the above

The change in KE is equal to the work done by the forces acting on the bead. The work done by gravity is –mg$R$, the work done by P is $P(\pi/2)R = mg\pi R$. Therefore $mv^2/2 = mg\pi R - mgR$

Answer: (b)

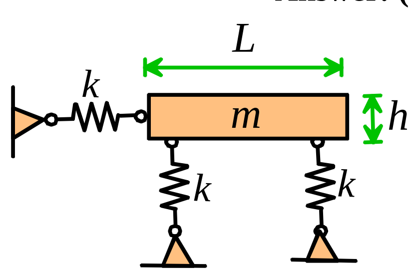
5. A bird lands near the tip of a branch, and is observed to oscillate up and down about once a second. The bird on the branch can be idealized as a lightly damped spring-mass system. When the vibration stops, the (static) deflection of the tip of the branch is approximately equal to

(a) 0 m
(b) 0.05 m
(c) 0.25 m
(d) 0.50 m
(e) 0.75 m
(f) 1 m

The bird and branch behave like a lightly damped spring-mass system. The static deflection is related to the natural frequency by $\omega_n = \sqrt{g/\delta} \Rightarrow \delta = g/\omega_n^2 = 9.8/(2\pi)^2 = 0.25m$

Answer: (c)

6. A vehicle mounted on its suspension system is idealized as a rigid body supported by three springs as shown in the figure. How many vibration frequencies does the system have, assuming that motion is confined to the plane of the figure?



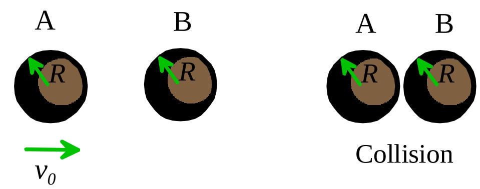
a) 2
b) 3
c) 4
d) 6
e) None of the above

The system has 3 DOF – motion horizontally, vertically and rotation. There are therefore 3 natural frequencies.

Answer (b)

7. The figure shows a collision between two identical spheres. The restitution coefficient for the collision *e=0*. Before the collision, A moves with speed v_0 and B is stationary. During the collision

(a) Momentum and energy are both conserved
(b) Momentum is conserved, and the energy increases
(c) Momentum is conserved, and the energy decreases
(d) Energy is conserved and momentum increases
(e) Energy is conserved and momentum decreases



Momentum is conserved during a collision. Since *e=0* the two spheres have the same velocity after collision – this means that after collision $v = v_0 / 2$. The total energy after collision is $2m(v_0/2)^2/2 = mv_0^2/4$. This is half the energy before collision.

Answer (c)

8. A ‘Critically Damped’ vibrating system

(a) Vibrates forever if it is disturbed from equilibrium
(b) Vibrates if disturbed from equilibrium but the vibrations decay quickly
(c) Returns to equilibrium following a disturbance without vibration
(d) Never returns to its equilibrium configuration if disturbed
(e) Feels wet and insulted.

The critically damped solution decays exponentially to equilibrium. There is no vibration.

Answer (c)

9. Beats are heard when two sounds have

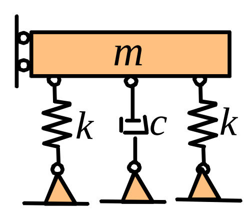
A) nearly the same amplitude
B) nearly the same frequencies
C) twice the amplitude
D) exactly twice the wavelength

Beats occur when two signals with similar frequencies combine – because of the difference in frequencies the signals alternately interfere constructively and destructively to give a slowly varying amplitude modulation.

Answer (b)

10. The viscous damping factor for the system shown in the figure is

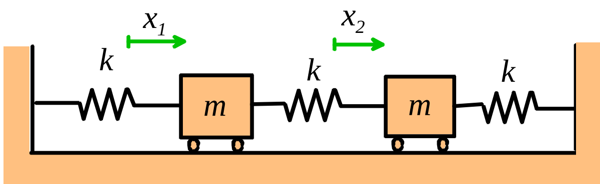
(a) $\zeta = c / \sqrt{2km}$
(b) $\zeta = c / 2\sqrt{2km}$
(c) $\zeta = 2c / \sqrt{2km}$
(d) $\zeta = c / 2\sqrt{km}$



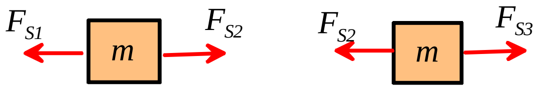
The two springs are in parallel, so the effective stiffness is $k_{eff} = 2k$. The standard formula for damping coefficient for a spring-mass system gives $\zeta = c / 2\sqrt{k_{eff} m} = c / 2\sqrt{2km}$

Answer (b)

11. The figure shows two identical masses that are connected to springs with stiffness k. The masses vibrate with displacements $x_1(t), x_2(t)$ from their equilibrium positions. When $x_1 = x_2 = 0$ there is no force in the springs.



11.1 Draw a free body diagram for each mass on the figure provided below



(3 POINTS)

11.2 Write down the changes in length of each of the three springs in terms of x_1, x_2. Hence, use Newton's laws to show that $x_1(t), x_2(t)$ satisfy the equations of motion

$$m\frac{d^2x_1}{dt^2} + 2kx_1 - kx_2 = 0$$

$$m\frac{d^2x_2}{dt^2} - kx_1 + 2kx_2 = 0$$

The increase in spring lengths (from left to right) are $x_1, x_2 - x_1, -x_2$

The spring forces follow as $F_{S1} = kx_1 \qquad F_{S2} = k(x_2 - x_1) \qquad F_{S3} = -kx_2$

Newton's law for the two masses gives

$$m\frac{d^2x_1}{dt^2} = -F_{S1} + F_{S2} = -kx_1 + k(x_2 - x_1)$$

$$m\frac{d^2x_2}{dt^2} = -F_{S2} + F_{S3} = -k(x_2 - x_1) - kx_2$$

These can be rearranged into the form given.

(3POINTS)

11.3 Add and subtract the equations of motion to show that the normal modes $q_1 = x_1 + x_2 \qquad q_2 = x_1 - x_2$ satisfy equations of the form

$$\frac{d^2q_1}{dt^2} + \omega_1^2 q_1 = 0 \qquad \frac{d^2q_2}{dt^2} + \omega_2^2 q_2 = 0$$

Hence, determine formulas for the two natural frequencies ω_1, ω_2.

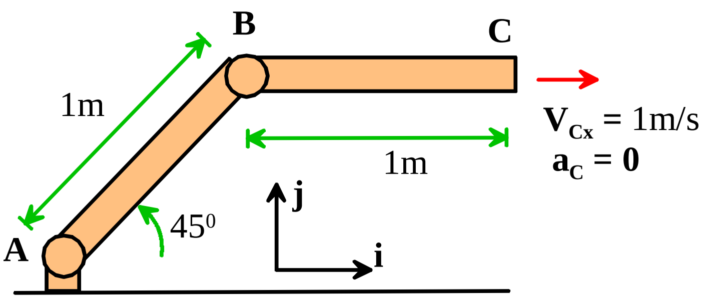
Adding the equations gives $m\frac{d^2}{dt^2}(x_1 + x_2) + k(x_1 + x_2) = 0 \Rightarrow \frac{d^2q_1}{dt^2} + \frac{k}{m}q_1 = 0$

Subtracting gives $m\frac{d^2}{dt^2}(x_1 - x_2) + 3k(x_1 - x_2) = 0 \Rightarrow \frac{d^2q_1}{dt^2} + \frac{3k}{m}q_1 = 0$

The natural frequencies follow as $\omega_1 = \sqrt{k/m} \qquad \omega_2 = \sqrt{3k/m}$

(4 POINTS)

12. The figure shows a robot arm. Point C on the arm is required to move horizontally with constant speed 1m/s. This is accomplished by rotating links AB and BC with appropriate angular speeds ω_{AB}, ω_{BC} and angular accelerations α_{AB}, α_{BC}. The goal of this problem is to calculate values for ω_{AB}, ω_{BC}, α_{AB}, α_{BC} at the instant shown.



12.1 Determine formulas for the velocity vectors $\mathbf{v}_B, \mathbf{v}_C$ of points B and C, in terms of ω_{AB}, ω_{BC}. (You do not need to solve for ω_{AB}, ω_{BC} until 12.3).

Applying the rigid body kinematics formula gives

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \omega_{AB}\mathbf{k} \times (\mathbf{i}+\mathbf{j})/\sqrt{2} = \omega_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2}$$

$$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC}\mathbf{k} \times \mathbf{i} = \omega_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2} + \omega_{BC}\mathbf{j}$$

(3 POINTS)

12.2 Determine formulas for the acceleration vectors $\mathbf{a}_B, \mathbf{a}_C$ of points B and C in terms of $\alpha_{AB}, \alpha_{BC}, \omega_{AB}, \omega_{BC}$. (You do not need to solve for α_{AB}, α_{BC} until 12.3)

$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \alpha_{AB}\mathbf{k} \times (\mathbf{i}+\mathbf{j})/\sqrt{2} + \omega_{AB}\mathbf{k} \times \omega_{AB}\mathbf{k} \times (\mathbf{i}+\mathbf{j})/\sqrt{2} \\ &= \alpha_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2} - \omega_{AB}^2(\mathbf{i}+\mathbf{j})/\sqrt{2}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_B + \alpha_{BC}\mathbf{k} \times \mathbf{i} + \omega_{BC}\mathbf{k} \times \omega_{BC}\mathbf{k} \times \mathbf{i} \\ &= \alpha_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2} - \omega_{AB}^2(\mathbf{i}+\mathbf{j})/\sqrt{2} + \alpha_{BC}\mathbf{j} - \omega_{BC}^2\mathbf{i}\end{aligned}$$

(3 POINTS)

12.3 Hence, calculate the required values of $\omega_{AB}, \omega_{BC}, \alpha_{AB}, \alpha_{BC}$

We know that $\mathbf{v}_C = \mathbf{i}$. Using the $\mathbf{i}$ and $\mathbf{j}$ components of 12.1 gives two equations for ω_{AB}, ω_{BC}

$$-\omega_{AB} = \sqrt{2} rad/s \qquad \omega_{AB}/\sqrt{2} + \omega_{BC} = 0 \Rightarrow \omega_{BC} = 1 rad/s$$

We also know that $\mathbf{a}_C = \mathbf{0}$ which gives

$$\alpha_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2} - \omega_{AB}^2(\mathbf{i}+\mathbf{j})/\sqrt{2} + \alpha_{BC}\mathbf{j} - \omega_{BC}^2\mathbf{i} = \mathbf{0}$$

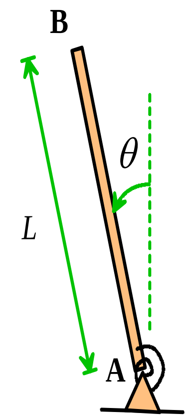
$$\Rightarrow \alpha_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2} + \alpha_{BC}\mathbf{j} - 2(\mathbf{i}+\mathbf{j})/\sqrt{2} - \mathbf{i} = \mathbf{0}$$

$$\Rightarrow \alpha_{AB} = -\left(2+\sqrt{2}\right) \; rad/s^2$$

$$\alpha_{BC} = \frac{1}{4\sqrt{2}} - \frac{\alpha_{AB}}{\sqrt{2}} = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(2+\sqrt{2}\right) = 1 + 2\sqrt{2} \, rad/s^2$$

(4 POINTS)

13. The figure shows a bar with mass m and length L that is pivoted about point A. The bar is stabilized by a torsional spring with stiffness κ, which exerts a restoring moment with magnitude $\kappa\theta$ at A. The goal of this problem is to determine the natural frequency of small amplitude vibrations of the bar.



13.1 State, or derive, the mass moment of inertia of the bar about point A, in terms of m and L.

The mass moment of inertia about the center is $mL^2/12$ (you can derive this from scratch as $I = \int_{-L/2}^{L/2} \frac{m}{L} x^2 dx = \frac{1}{12}\frac{m}{L}L^3 = \frac{1}{12}mL^2$)

The parallel axis theorem then gives $I_A = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$

(2 POINTS)

13.2 Write down the total potential energy of the system, in terms of m, g, L, κ, θ.

The potential energy is $mg\frac{L}{2}\cos\theta + \frac{1}{2}\kappa\theta^2$

(2 POINTS)

13.3 Write down the total kinetic energy of the system, in terms of $m, L, \frac{d\theta}{dt}$.

The kinetic energy is $\frac{1}{2}I_A\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{6}mL^2\left(\frac{d\theta}{dt}\right)^2$

(2 POINTS)

13.4 Hence, use energy conservation to show that θ satisfies

$$\frac{mL^2}{3}\frac{d^2\theta}{dt^2} + \kappa\theta - mg\frac{L}{2}\sin\theta = 0$$

The total energy is constant so $\frac{1}{6}mL^2\left(\frac{d\theta}{dt}\right)^2 + mg\frac{L}{2}\cos\theta + \frac{1}{2}\kappa\theta^2 = C$

Differentiate with respect to time

$$\frac{1}{3}mL^2\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} - mg\frac{L}{2}\sin\theta\frac{d\theta}{dt} + \kappa\theta\frac{d\theta}{dt} = 0$$

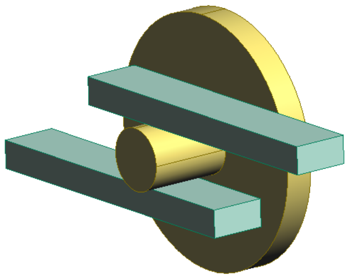
$$\Rightarrow \frac{1}{3}mL^2\frac{d^2\theta}{dt^2} - mg\frac{L}{2}\sin\theta + \kappa\theta = 0$$

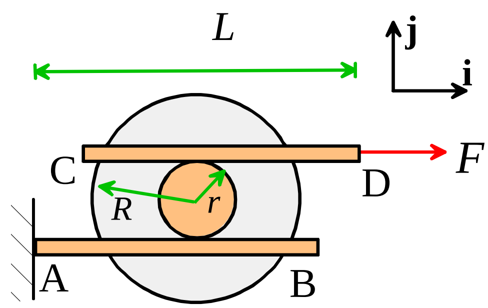
(2 POINTS)

13.5 Finally, determine a formula for the natural frequency of vibration.

Recall that $\sin\theta \approx \theta \Rightarrow \frac{d^2\theta}{dt^2} + 3\left(\frac{\kappa}{mL^2} - \frac{g}{2L}\right)\theta = 0 \Rightarrow \omega_n = \sqrt{3\left(\frac{\kappa}{mL^2} - \frac{g}{2L}\right)}$

(2 POINTS)



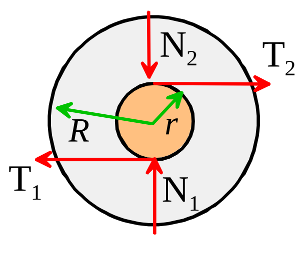


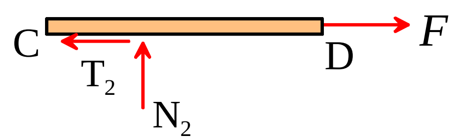
14. An ‘inerter’ is a suspension element that exerts a force *F* that is related to its length *L* by

$$F = \mu \frac{d^2 L}{dt^2}$$

The figure shows a proposed design for such a device. It consists of a disk with mass *m*, radius *R* and mass moment of inertia $mR^2/2$ which is rigidly connected to an axle with radius *r*. The axle rolls without slip between platens AB and CD (which have negligible mass). The objective of this problem is to derive an equation for the coefficient μ

14.1 Draw a free body diagram showing the forces acting on the disk/axle and platen CD on the figures provided below. Gravity may be neglected.

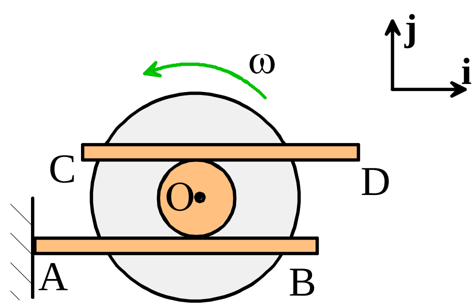




(The normal forces are optional)

(3 POINTS)

14.2 Suppose AB is stationary and the disk and axle rotate with angular velocity $\omega\mathbf{k}$. Find the velocity vectors $\mathbf{v}_O$ and $\mathbf{v}_D$ of the center O of the axle and point D in terms of *r* and ω



The wheel rolls without slip on the member AB. The center therefore has velocity $\mathbf{v}_O = -\omega r\mathbf{i}$. There is no slip at the contact between the axle and CD. D therefore moves with the same velocity as the axle at the contact point – this is $\mathbf{v}_D = -2\omega r\mathbf{i}$

(2 POINTS)

14.3 Write down the equations of linear (**F**=m**a**) and rotational ($\mathbf{M} = I\boldsymbol{\alpha}$) motion for the disk.

Newton's law gives $(T_2 - T_1)\mathbf{i} + (N_1 - N_2)\mathbf{j} = ma_O\mathbf{i}$ (the **j** component should be consistent with FBD)
The rotational equation gives $-(T_1 + T_2)r\mathbf{k} = I\alpha\mathbf{k}$

(2 POINTS)

14.4 Hence, show that

$$\mu = \frac{m}{4}(1 + \frac{R^2}{2r^2})$$

From problem 14.2
$v_D = -2r\omega \Rightarrow a_D = -2r\alpha \Rightarrow \alpha = -a_D / (2r)$
$v_0 = -r\omega \Rightarrow a_0 = -r\alpha = a_D / 2$

From problem 14.3

$$T_2 - T_1 = -mr\alpha = ma_D / 2$$
$$T_2 + T_1 = -I\alpha / r = a_D I / (2r^2)$$
$$\Rightarrow 2T_2 = \frac{1}{2}a_D(m + I / r^2)$$

Finally the FBD for the bar shows that $T_2 = F$ and of course $a_D = \dfrac{d^2L}{dt^2}$. Therefore (substituting for I)

$$F = \frac{m}{4}(1 + \frac{R^2}{2r^2})\frac{d^2L}{dt^2}$$

(3 POINTS)