School of Engineering
Brown University

EN40: Dynamics and Vibrations

Midterm Examination
Tuesday March 5 2013

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

____

1 (10 points) _____________

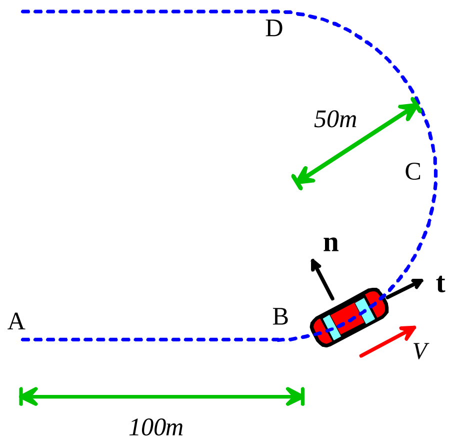
2. (8 points) _____________

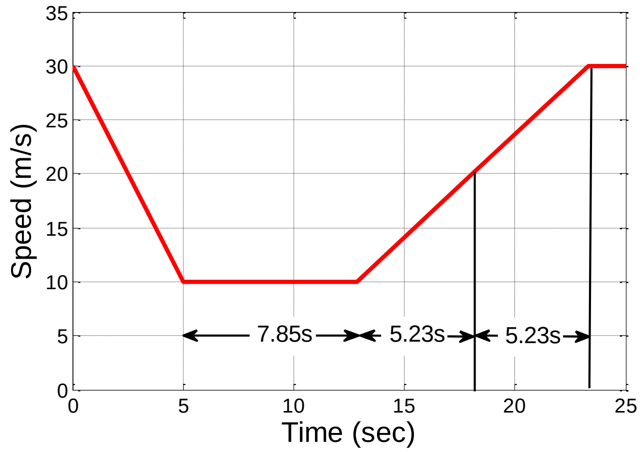
3. (8 points) _____________

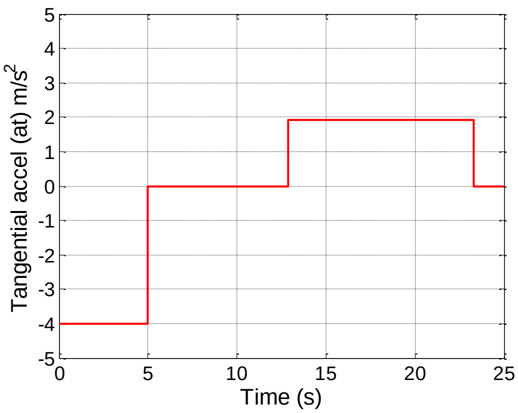
4. (8 points) _____________

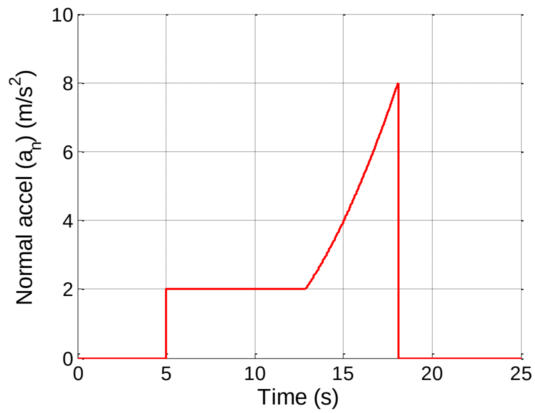
TOTAL (34 points)_____________

3. The path and speed of a vehicle driving around a sharp bend is shown in the figure below (the vehicle is at point A at time $t=0$). Sketch graphs of the normal and tangential acceleration of the vehicle on the axes provided. Explain briefly how you calculated relevant quantities.









[1 POINT] **[1 POINT]**

This problem involves applying and interpreting the formula for acceleration in normal-tangential coordinates

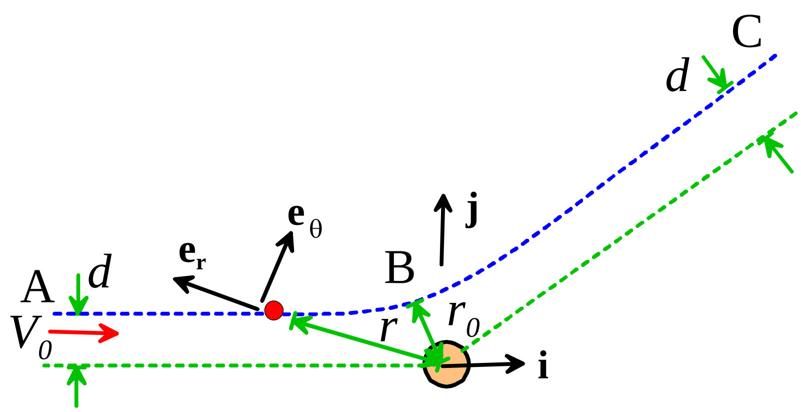
$$\mathbf{a} = \frac{dV}{dt}\mathbf{t} + \frac{V^2}{R}\mathbf{n}$$

- During the first 5 sec the car travels with average speed 20 m/s, and therefore travels a total distance 100m and just reaches B. [**1 POINT]**
- During the first 5 sec the tangential acceleration is -20/5=-4m/s^2 **[1 POINT]**
- The distance traveled during the subsequent 7.85 sec is 78.5m – the angle BOC is therefore 78.5/50=1.57 radians (approximately $\pi/2$). The car reaches C at the end of this time. **[1 POINT]**
- During this period the tangential acceleration is zero (speed is constant). The normal acceleration is 100/50=2m/s^2 **[1 POINT]**
- During the subsequent 5.23sec, the average speed is 15m/s, and the car therefore travels 78.5m and reaches D at the end of the time period. **[1 POINT]**
- The tangential acceleration is equal to 10/5.23 = 1.91m/s^2 (for 10.46 sec) **[1 POINT]**
- The speed during this phase varies linearly with time – the normal acceleration varies quadratically. At D, the speed is 20 m/s, so the normal acceleration is 400/50=8 m/s^2. **[1 POINT]**
- Thereafter, the normal acceleration drops to zero (the radius of curvature is infinity). **[1 POINT]**

2. In a Rutherford scattering experiment, a positively charged alpha particle with mass m is fired with speed $V_0\mathbf{i}$ from a distant point towards a stationary, positively charged nucleus. The charged particle experiences a radial repulsive force

$$\mathbf{F} = \frac{K}{r^2}\mathbf{e}_r$$

where K is a constant. The trajectory of the particle is sketched in the figure below.



2.1 Find an expression for the potential energy of the force acting on the alpha particle, in terms of K and r.

The potential energy is $V = -\int_{\infty}^{r} \frac{K}{r^2} dr = \frac{K}{r}$

[2 POINTS]

2.2 Use energy conservation to show that the speed of the particle v is related to its distance from the nucleus r by

$$v^2 = V_0^2 - \frac{2K}{mr}$$

Energy conservation gives

$$T_0 + V_0 = T + V \Rightarrow \frac{1}{2}mV_0^2 = \frac{1}{2}mv^2 + \frac{K}{r}$$

This can be re-arranged into the form stated.

[1 POINT]

2.3 Write down the angular momentum vector of the particle about the origin at time *t=0*. (Don't forget to specify the direction as well as the magnitude)

$$\mathbf{h}_0 = \mathbf{r} \times m\mathbf{v} = -mdV_0\mathbf{k}$$

[2 POINTS]

2.4 Hence, show that the minimum distance of the particle from the nucleus is

$$r_0 = \frac{K}{mV_0^2}\left[1+\sqrt{1+\left(\frac{V_0^2 md}{K}\right)^2}\right]$$

At the minimum distance, the radial component of velocity must be zero (since dr/dt=0 at the minimum value of *r*). This means that the velocity vector is perpendicular to the position vector, and hence the angular momentum is

$$\mathbf{h} = -mvr_0\mathbf{k}$$

Angular momentum conservation thus gives

$$vr_0 = dV_0 \Rightarrow v = \frac{d}{r_0}V_0$$

Energy conservation (from 2.2) already showed that

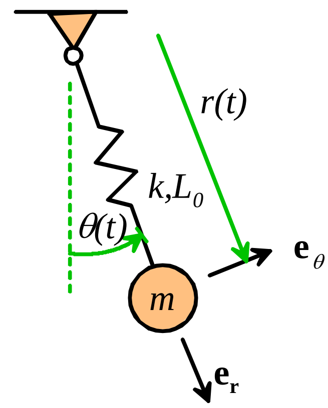
$$v^2 = V_0^2 - \frac{2K}{mr} \Rightarrow \left(\frac{d}{r_0}V_0\right)^2 = V_0^2 - \frac{2K}{mr_0}$$

$$\Rightarrow r_0^2 - \frac{2K}{mV_0^2}r_0 - d^2 = 0$$

$$\Rightarrow r_0 = \frac{K}{mV_0^2} + \frac{1}{2}\sqrt{\left(\frac{2K}{mV_0^2}\right)^2 + 4d^2} = \frac{K}{mV_0^2}\left[1+\sqrt{1+\left(\frac{mdV_0^2}{K}\right)^2}\right]$$

[**3 POINTS]**

3. The figure shows a flexible pendulum, idealized as a point mass m at the end of a spring with stiffness k and un-stretched length L_0.



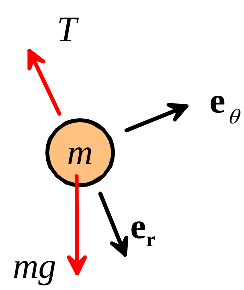
3.1 Write down the acceleration of the mass, expressed as components in the polar-coordinate basis $\{\mathbf{e}_r, \mathbf{e}_\theta\}$

We can use the formula from notes:

$$\mathbf{a} = \left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} - 2\frac{dr}{dt}\frac{d\theta}{dt} \right)\mathbf{e}_\theta$$

[2 POINTS]

3.2 Draw the forces acting on the mass on the figure provided.



[2 POINTS]

3.3 Hence, show that r, θ satisfy the differential equations

$$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - g\cos\theta + \frac{k}{m}(r - L_0) = 0 \qquad r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} + g\sin\theta = 0$$

The spring force gives $T = k(r - L_0)$.

Newton's law gives

$$\mathbf{F} = m\mathbf{a} = m\left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right)\mathbf{e}_r + m\left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right)\mathbf{e}_\theta = (mg\cos\theta - T)\mathbf{e}_r - mg\sin\theta\,\mathbf{e}_\theta$$

Combining these, the two components of Newton's law give the expressions stated.

[2 POINTS]

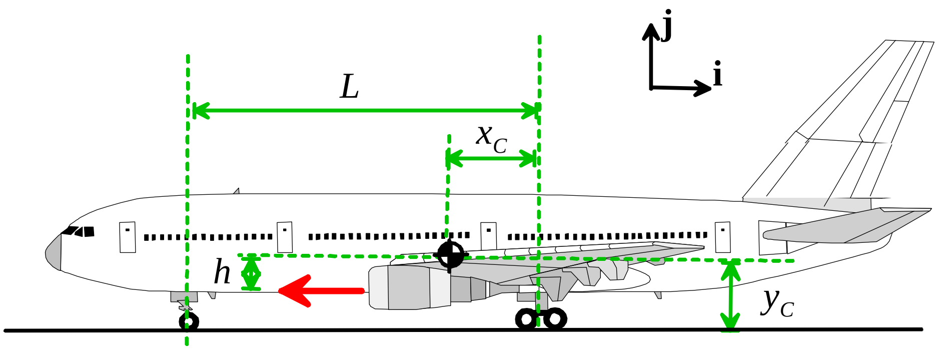
3.4 Re-write the equations into a form that MATLAB could solve

As always, we must introduce $v = \frac{dr}{dt}$ $\omega = \frac{d\theta}{dt}$ as additional variables. The equations thus reduce to

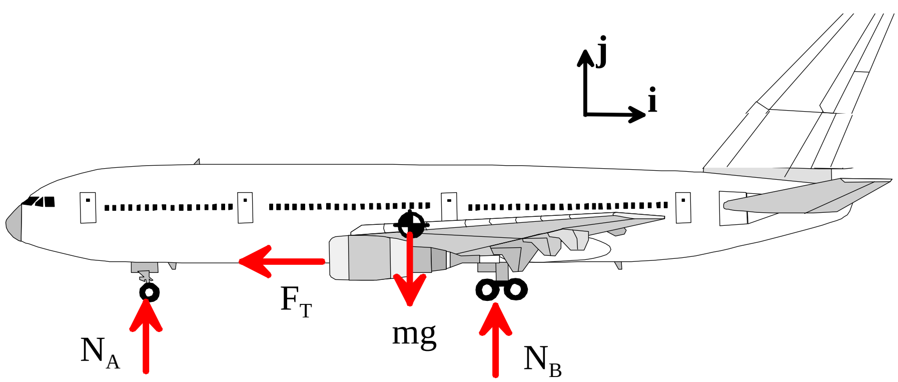
$$\frac{d}{dt}\begin{bmatrix} r \\ \theta \\ v \\ \omega \end{bmatrix} = \begin{bmatrix} v \\ \omega \\ r\left(\frac{d\theta}{dt}\right)^2 + g\cos\theta + \frac{k}{m}(r - L_0) \\ -\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} - \frac{g}{r}\sin\theta \end{bmatrix}$$

[2 POINTS]

4. The figure shows an aircraft just starting its take-off roll. The engines provide a total thrust F_T that act a height h below the center of mass, producing an acceleration $a_x = -(g/2)\mathbf{i}$. Since the aircraft is not yet moving lift and drag forces are zero.



4.1 Draw the forces acting on the aircraft on the figure provided below.



[2 POINTS]

4.2 Write down Newton's law of motion and the equation of rotational motion for the aircraft (assume straight line motion without rotation)

$$\mathbf{F} = m\mathbf{a} \Rightarrow -F_T\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = -\frac{g}{2}\mathbf{i}$$
$$\mathbf{M}_C = \left[-N_A(L - x_c) - F_T h + x_c N_B\right]\mathbf{k} = \mathbf{0}$$

[2 POINTS]

4.3 Hence, find formulas for the reaction forces on the wheels (for the rear wheels, calculate the total force).

The previous problem gives three equations, which can be solved for N_A, N_B, F_T with the results

$$F_T = mg/2$$
$$N_A = mg\left(\frac{x_c}{L} - \frac{h}{2L}\right)$$
$$N_B = mg\left(1 + \frac{h}{2L} - \frac{x_c}{L}\right)$$

[2 POINTS]

4.4 Hence, show that the front wheel will lose contact with the ground if *h* exceeds a critical value, and find a formula for this critical value of *h*.

Notice that $N_A > 0 \Leftrightarrow \left(\frac{x_c}{L} - \frac{h}{2L}\right) > 0 \Rightarrow h < 2x_c$

The critical value of *h* is $2x_c$

[2 POINTS]