School of Engineering
Brown University

EN40: Dynamics and Vibrations

Final Examination
Monday May 13 2013: 2pm-5pm

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

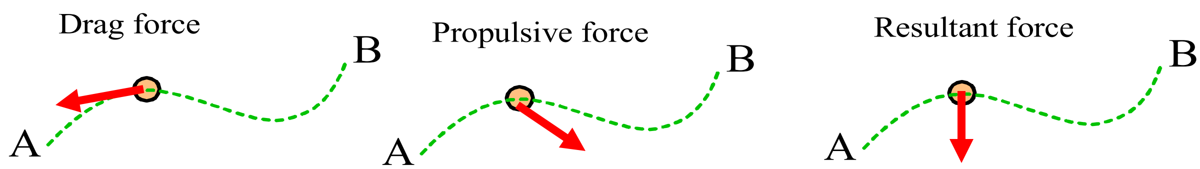
`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

____

1-8 [20 points]	______________
9 [7 POINTS]	______________
10 [13 POINTS]	______________
11 [10 POINTS]	______________
12 [10 POINTS]	______________
TOTAL [60 POINTS]	______________

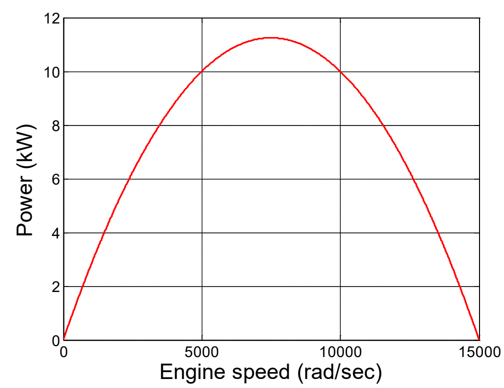
FOR PROBLEMS 1-8 WRITE YOUR ANSWER IN THE SPACE PROVIDED. ONLY THE ANSWER APPEARING IN THE SPACE PROVIDED WILL BE GRADED. ILLEGIBLE ANSWERS WILL NOT RECEIVE CREDIT.

1. The figure shows the path of a predator in the predator-prey competition, which travels along the dashed line from A to B. At the instant shown, the predator travels at constant speed with velocity **v**. It is subjected to a viscous drag force $\mathbf{F}_D = -c\mathbf{v}$ and the random force is zero. On the figures below, draw arrows showing the directions of (a) the viscous drag force; (b) the propulsive force; and (c) the resultant force.



(2 POINTS)

2. The figure shows a power curve (with power in kW and engine speed in rad/s) for an internal combustion engine. It powers a vehicle with mass 1000kg that runs at constant speed up a 5% grade (i.e. $\sin(\theta) \approx 1/20$). The engine runs at 10000 rad/s. If transmission losses and air resistance can be neglected, the speed of the vehicle is



(a) 10 m/s
(b) 15 m/s
(c) 20 m/s
(d) 25 m/s
(e) none of the above

The power at 10000 rad/s is 10kW. The power is related to speed by

$$P = mgv\sin\theta \Rightarrow 1000 \times 10 \times v \times 1/20 = 10000 \Rightarrow v = 20m/s$$

ANSWER______C________ **(2 POINTS)**

3. The vehicle described in the preceding problem has a wheel radius of 0.5m. The transmission has gear ratio (motor angular speed/axle angular speed)

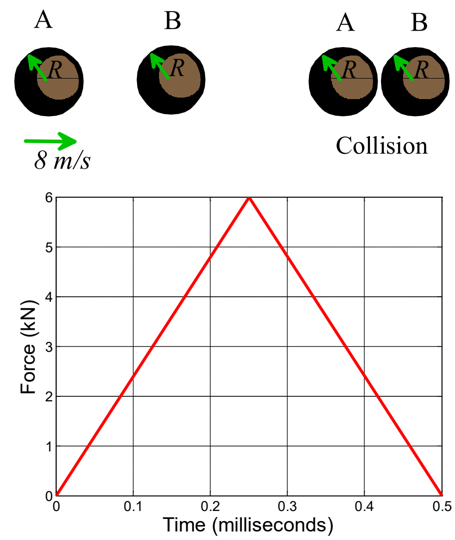
(a) 400
(b) 250
(c) 1/250
(d) 1/400
(e) None of the above

The axle angular speed is related to the vehicle speed by

$$v = \omega_a R_w \Rightarrow 20 = \omega_a \times 0.5 \Rightarrow \omega_a = 40$$
$$\Rightarrow \omega_m / \omega_a = 10000/40 = 250$$

ANSWER______B________ **(2 POINTS)**

4. The figure shows the variation of impact force with time during the straight-line collision between two identical spheres with 0.25kg mass. Before the collision, one sphere is stationary; the other has speed 8m/s.



4.1 The impulse of the force is

(a) 1 Ns
(b) 1.5 Ns
(c) 2 Ns
(d) 3 Ns
(e) None of the above

The impulse of the force is

$$I = \frac{1}{2} 6000 \times 0.5 \times 10^{-3} = 1.5 Ns$$

ANSWER______B_______ **(2 POINTS)**

4.2 The velocities of the spheres after impact are

(a) $V_{A1} = 2m/s \quad V_{B1} = 6m/s$
(b) $V_{A1} = 0m/s \quad V_{B1} = 8m/s$
(c) $V_{A1} = 4m/s \quad V_{B1} = 4m/s$
(d) $V_{A1} = 1m/s \quad V_{B1} = 7m/s$
(e) None of the above

After impact, the two spheres have speeds

$$V_{A1} = V_{A0} - I/m = 8 - 1.5/0.25 = 2m/s$$

$$V_{B1} = I/m = 1.5/0.25 = 6m/s$$

ANSWER______A________ **(2 POINTS)**

4.3 The restitution coefficient for the collision is

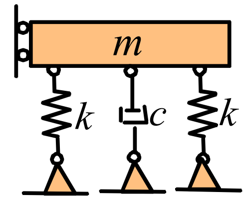
(a) e=0
(b) e=0.25
(c) e=0.5
(d) e=0.75
(e) e=1

The restitution coefficient follows as

$$e = \frac{V_{B1} - V_{A1}}{V_{A0}} = \frac{6-2}{8} = 0.5$$

ANSWER_______C_______ **(2 POINTS)**

5. The spring-mass system shown in the figure is critically damped. If one spring is removed, the damping factor is



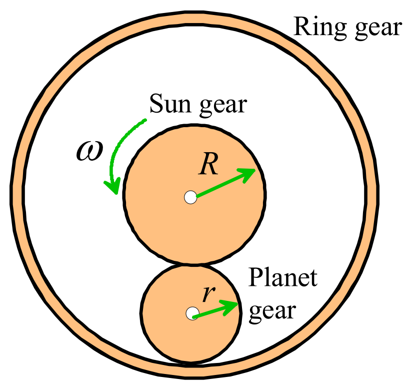
(a) $\zeta = 1$
(b) $\zeta = 2$
(c) $\zeta = \sqrt{2}$
(d) $\zeta = 1/\sqrt{2}$

(e) $\zeta = 1/2$

The original system has damping factor $\zeta = 1/2\sqrt{2km}$ and for critically damped systems $\zeta = 1$
After the modification, the damping factor changes to $\zeta = 1/2\sqrt{km}$. Therefore $\zeta = \sqrt{2}$.

ANSWER_____C_________ **(2 POINTS)**

6. In the figure shown, the ring gear is stationary and the sun gear rotates in the counter-clockwise direction with angular speed ω. The angular speed of the planet gear is



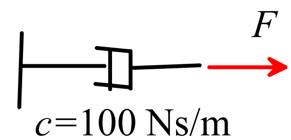
(a) $\omega_p = \omega R / r$ counterclockwise
(b) $\omega_p = \omega R / r$ clockwise
(c) $\omega_p = \omega R / (2r)$ counterclockwise
(d) $\omega_p = \omega R / (2r)$ clockwise

(e) None of the above

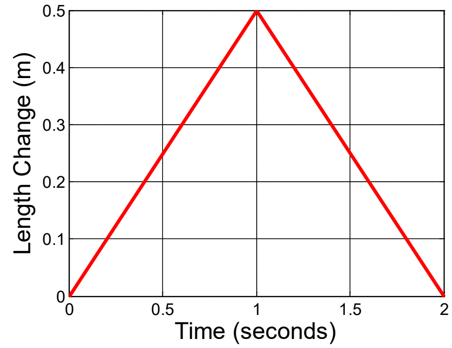
The planet gear is stationary where it touches the ring gear, and moves with the same speed as the sun gear where it touches the sun gear. By inspection its rotation will be in the opposite direction to the sun gear. Therefore $\omega_p = \omega R / (2r)$ clockwise

ANSWER_______D_______ **(2 POINTS)**

7. The dashpot shown in the figure has dashpot coefficient $c = 100 Ns/m$. The end of the dashpot is subjected to a force that causes its length to vary as indicated in the graph. The total work done by the force after 2 sec is



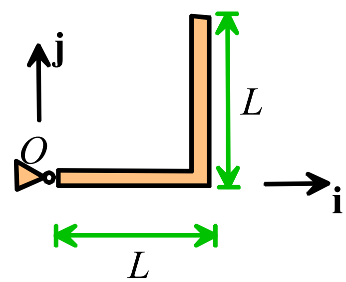
(a) 0
(b) 12.5 J
(c) 25 J
(d) 50 J
(e) None of the above



The extension rate is 0.5m/s. The magnitude of the force is therefore $F = 100 \times 0.5 = 50N$ and the power developed is $F\dfrac{dL}{dt} = 25W$. The total work done is therefore $2 \times 25 = 50J$.

ANSWER______D________ **(2 POINTS)**

8. The mass moment of inertia about the center of mass of a slender rod with mass m length L is $I_G = mL^2/12$. Two slender rods with mass m are rigidly connected to create a right-angle section as shown in the figure. The mass moment of inertia of the right angle section **about the k axis through point O** is

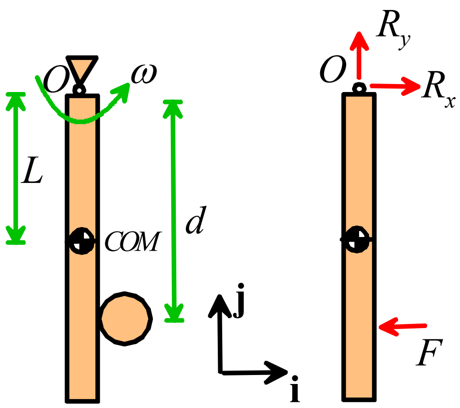


(a) $I_O = 2mL^2/3$
(b) $I_O = 17mL^2/12$
(c) $I_O = 5mL^2/3$
(d) $I_O = mL^2/6$
(e) None of the above

$$\frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 + \frac{1}{12}mL^2 + m\left[\left(\frac{L}{2}\right)^2 + L^2\right] = \frac{5}{3}mL^2$$

ANSWER_______C_______ **(2 POINTS)**

9. The figure shows the forces acting on a baseball bat at the instant when it strikes the baseball (in plan view). The reaction forces R_x, R_y are exerted by the batter's hands; the force F is exerted by the ball. The bat has mass m and mass moment of inertia $I_O = \frac{4}{3}mL^2$ ***about the point O (NOT the COM)*** where L is the distance of the center of mass of the bat from the handle. At the instant shown, the bat rotates in the horizontal plane about the batter's grip (at O) with instantaneous angular speed ω and acceleration α.



9.1 Using the rigid body kinematics formulas, find a formula for the acceleration of the center of mass of the bat in terms of L, ω and α, expressing your answer in $\{\mathbf{i},\mathbf{j}\}$ components. You do not need to use Newton's laws to answer this part.

$$\begin{aligned}\mathbf{a}_G &= \mathbf{a}_O + \alpha\mathbf{k}\times\mathbf{r}_{G/O} + \omega\mathbf{k}\times\omega\mathbf{k}\times\mathbf{r}_{G/O} \\ &= \mathbf{0} + \alpha L\mathbf{i} + \omega^2 L\mathbf{j}\end{aligned}$$

[2 POINTS]

9.2 Using Newton's laws, the equation of rotational motion, and the solution to 10.1, find formulas for the reaction forces R_x, R_y in terms of *m, L, d* ω and the impact force *F*.

Newton's laws and rotational motion around O gives

$$\begin{aligned}&(R_x - F)\mathbf{i} + R_y\mathbf{j} = m\alpha L\mathbf{i} + m\omega^2 L\mathbf{j} \\ &I_0\alpha\mathbf{k} = -Fd\mathbf{k} \\ &I_0 = \frac{4}{3}mL^2 \\ &\Rightarrow R_x = F\left(1 - \frac{3mdL}{4mL^2}\right) \qquad R_y = m\omega^2 L\end{aligned}$$

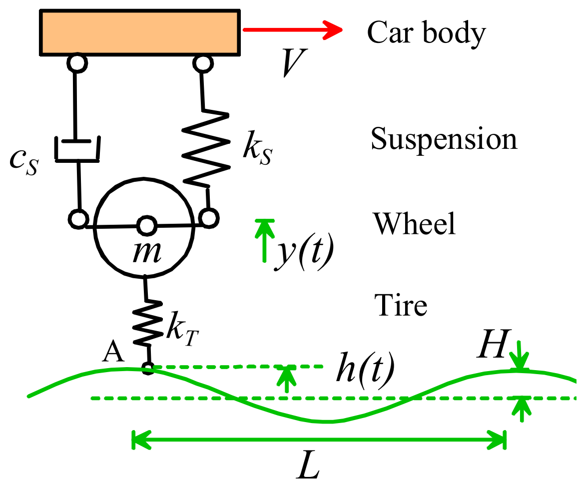
[3 POINTS]

9.3 Find a formula for the distance d that will minimize the magnitude of the reaction force at O.

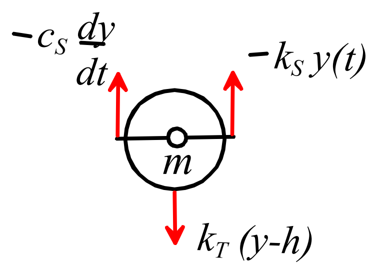
The magnitude is minimized if $R_x = 0 \Rightarrow d = 4L/3$

[2 POINTS]

10. 'Rumble strips' are roughened regions on a road that warn drivers when their vehicle approaches the side of a highway. The figure shows an idealization of one wheel of a vehicle driving over a 'rumble strip' with amplitude H and wavelength L. The mass represents the wheel; the spring with stiffness k_T models the deformation of the tire, and the combined spring and damper k_S, c_S represent the car's suspension. The vehicle travels with steady horizontal speed V. As a result, the contact point of the tire with the road at A experiences a vertical harmonic motion $h(t) = H \sin \omega t$, where $\omega = 2\pi V / L$. The tire vibrates vertically with a displacement from its static equilibrium position *y(t)*. Vertical motion of the car body can be neglected.



10.1 Draw a free body diagram showing all the time dependent forces acting on the wheel. **Gravitational forces may be neglected throughout this problem since they do not vary with time.**



[3 POINTS]

10.2 Hence, show that the equation of motion for the tire displacement *y(t)* is

$$m\frac{d^2y}{dt^2} + c_S\frac{dy}{dt} + (k_T + k_S)y = k_T H \sin \omega t$$

Newton's law gives $m\frac{d^2y}{dt^2} = -c_S\frac{dy}{dt} - k_T(y-h) - k_S y$. Substituting for h and re-arranging gives the expression stated.

[3 POINTS]

10.3 Re-write the equation of motion in standard form and find expressions for the natural frequency and damping coefficient ω_n, ζ in terms of m, k_S, k_T, c_S

$$\frac{m}{(k_T + k_S)}\frac{d^2 y}{dt^2} + \frac{c_S}{(k_T + k_S)}\frac{dy}{dt} + y = \frac{k_T H}{(k_T + k_S)}\sin \omega t$$

The standard form is

$$\frac{1}{\omega_n^2}\frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = KF_0 \sin \omega t$$

Our expression is the same as the standard formula if we identify

$$K = \frac{k_T}{k_T + k_S} \qquad F_0 = H \qquad \omega_n = \sqrt{\frac{k_T + k_S}{m}} \qquad \zeta = \frac{c_s}{2\sqrt{(k_T + k_S)m}}$$

[2 POINTS]

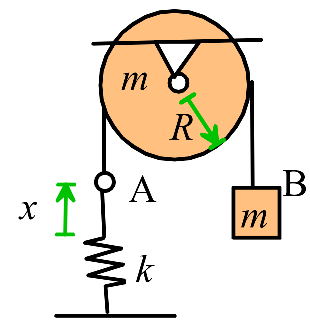
10.4 Find an expression for the amplitude of vibration of the wheel, in terms of $\omega, \omega_n, \zeta, k_T, k_S$ and *H*.

From the formulas given in class, the amplitude is

$$Y_0 = H\left(\frac{k_T}{k_T + k_S}\right)\frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\zeta^2 \frac{\omega^2}{\omega_n^2}}}$$

[2 POINTS]

11. The ends A and B of an inextensible cable are attached to a spring and mass, as shown in the figure. The cable passes over a pulley with mass m, radius R and mass moment of inertia $mR^2/2$.



11.1 Let x denote the extension of the spring (i.e. the difference between the spring's length and its unstretched length). Write down an expression for the total potential energy of the system in terms of k,x,m,g.

$$V = \frac{1}{2}kx^2 - mgx$$

[2 POINTS]

11.2 Assuming that the cable does not slip on the pulley, find an expression for the kinetic energy of the system, in terms of m and dx/dt

$$T = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}I\omega^2 \qquad I = mR^2/2 \qquad \omega = \frac{1}{R}\left(\frac{dx}{dt}\right)$$

$$\Rightarrow T = \frac{1}{2}\frac{3m}{2}\left(\frac{dx}{dt}\right)^2$$

[3 POINTS]

11.3 Hence, show that x satisfies the equation of motion

$$\frac{3}{2}m\frac{d^2x}{dt^2} + kx = mg$$

Energy is conserved so

$$T + V = C \Rightarrow \frac{d}{dt}\left(\frac{1}{2}\frac{3m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 - mgx\right) = 0$$

$$\Rightarrow \left(\frac{3m}{2}\frac{d^2x}{dt^2} + kx - mg\right)\frac{dx}{dt} = 0$$

This reduces to the expression stated.

[3 POINTS]

11.4 Find an expression for the natural frequency of vibration of the system in terms of k and m.

Rearrange the equation in standard form

$$\frac{3}{2}\frac{m}{k}\frac{d^2x}{dt^2}+(x-\frac{mg}{k})=0 \quad \Rightarrow \frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+y=0 \quad \omega_n=\sqrt{\frac{2k}{3m}} \quad y=x-mg$$

The natural frequency is therefore $\omega_n=\sqrt{\frac{2k}{3m}}$

[2 POINTS]

11.5 The system is released from rest with $x=0$. Find an expression for the subsequent maximum velocity of the mass attached to the cable at B.

We can solve this using vibrations – note that the velocity of the mass is dx/dt so we can solve the EOM for y and hence deduce dx/dt as follows

$$y=-\frac{mg}{k}\Rightarrow y=-\frac{mg}{k}\cos\omega_n t\Rightarrow \frac{dy}{dt}=\frac{dx}{dt}=\frac{mg}{k}\sqrt{\frac{2k}{3m}}\sin\omega_n t$$

The maximum value is thus $g\sqrt{\frac{2m}{3k}}$

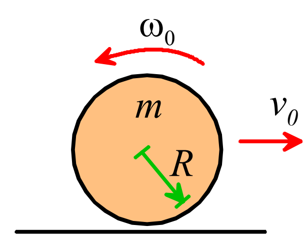
Alternatively if we didn't learn any vibrations ☹ we can note that $\frac{d^2x}{dt^2}=0$ when the velocity is a maximum, and so the EOM tells us that $kx=mg$ at the instant of max velocity. We can get the velocity when x has this value using energy conservation

$$T_0+V_0=0=T_1+V_1=\frac{1}{2}kx^2-mgx+\frac{1}{2}\frac{3m}{2}v^2$$

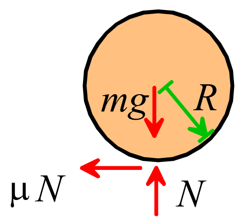
$$\Rightarrow v^2=-\frac{2}{3}\left(\frac{k}{m}x^2-2gx\right)=-\frac{2}{3}\left(\frac{k}{m}\left(\frac{mg}{k}\right)^2-2g\frac{mg}{k}\right)=\frac{2g^2m}{3k}$$

[3 POINTS]

12. A thin uniform disk of radius R, mass m and mass moment of inertia $mR^2/2$ is placed on the ground with a positive velocity v_0 in the horizontal direction, and a counterclockwise rotational velocity (a backspin) ω_0. The contact between the disk and the ground has friction coefficient μ. The disk initially slips on the ground, and for a suitable range of values of ω_0 and v_0 its direction of motion may reverse.



12.1 Draw a free body diagram showing the forces acting on the disk just after it hits the ground.



[3 POINTS]

12.2 Hence, find formulas for the initial acceleration a and angular acceleration α for the disk, in terms of g, R and μ. Note that the contact point is slipping.

Newton's law and the rotational equation give

$$N = mg \qquad \mu N = -ma_x \qquad \frac{1}{2}mR^2\alpha = -\mu NR$$
$$\Rightarrow a = -\mu g \qquad \alpha = -2\mu g / R$$

[2 POINTS]

12.3 Find a formula for the time at which the disk will reverse its direction of motion.

Velocity is reversed where $v=0$. The constant acceleration straight line motion formula gives

$v = v_0 - \mu g t \Rightarrow t = v_0 / \mu g$ at the reversal.

[2 POINTS]

12.4 Find a formula for the time at which the disk begins to roll on the ground without slip. Hence, show that the disk will reverse its direction only if $v_0 < \omega_0 R / 2$

Rolling without sliding starts when $v = -\omega R$. We have that

$$\omega = \omega_0 - 2\mu g t / R \qquad v = v_0 - \mu g t$$
$$\Rightarrow v_0 - \mu g t = -(\omega_0 R - 2\mu g t)$$
$$\Rightarrow t = (v_0 + \omega_0 R) / 3\mu g$$

The reversal will only occur if slip continues long enough – this means that

$(v_0 + \omega_0 R) / 3\mu g > v_0 / \mu g \Rightarrow v_0 < \omega_0 R / 2$

[3 POINTS]