School of Engineering
Brown University

EN40: Dynamics and Vibrations

Midterm Examination
Tuesday March 4 2014

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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1 (10 points) _____________

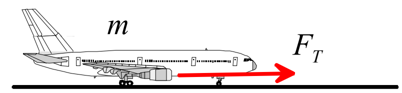
2. (13 points) _____________

3. (7 points) _____________

4. (10 points) _____________

TOTAL (40 points)___________

1. The engines on an aircraft generate a thrust that decreases with the aircraft speed according to the relation



$$F_T = F_0\left(1-\frac{v}{v_0}\right)$$

where F_0, v_0 are constants. The aircraft has mass *m*. It starts at rest at time *t=0* and must reach speed v_{TO} in order to take off.

1.1 Use Newton's law to determine the acceleration of the aircraft and hence determine an expression for its speed as a function of time and other parameters. Air resistance and friction may be neglected.

Newton gives $\frac{dv}{dt} = a_x = \frac{F_0}{m}\left(1-\frac{v}{v_0}\right)$

Separate variables and integrate

$$\int_0^v \frac{dv}{1-v/v_0} = \int_0^t \frac{F_0}{m}dt \Rightarrow -v_0\log(1-v/v_0) = \frac{F_0}{m}t \Rightarrow v = v_0\left(1-\exp\left\{-\frac{F_0}{mv_0}t\right\}\right)$$

[3 POINTS]

1.2 Find a formula for the distance traveled by the aircraft as a function of time.

$$\frac{dx}{dt} = v_0\left(1-\exp\left\{-\frac{F_0}{mv_0}t\right\}\right)$$

$$\Rightarrow \int_0^x dx = \int_0^t v_0\left(1-\exp\left\{-\frac{F_0}{mv_0}t\right\}\right)$$

$$\Rightarrow x = v_0 t + \frac{mv_0^2}{F_0}\left(\exp\left\{-\frac{F_0}{mv_0}t\right\}-1\right)$$

[3 POINTS]

1.3 Hence find a formula for the minimum length of runway necessary for the aircraft to reach take-off speed, in terms of F_0, v_0, *m*, and v_{TO}

The aircraft must reach take-off speed, so

$$-\frac{mv_0}{F_0}\log(1-v_{To}/v_0) = t$$

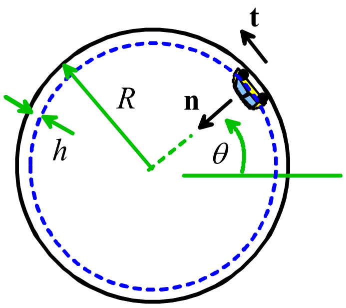
$$\Rightarrow -\frac{v_{T0}}{v_0} = \left(\exp\left\{-\frac{F_0}{mv_0}t\right\}-1\right)$$

and hence

$$d = v_0 t + \frac{mv_0}{F_0}\left(\exp\left\{-\frac{F_0}{mv_0}t\right\}-1\right) = -\frac{mv_0^2}{F_0}\log(1-v_{To}/v_0) - \frac{mv_0 v_{TO}}{F_0}$$

[4 POINTS]

2. The figure shows a picture of a car inside the 'Dunlop Death Loop.' The goal of this problem is to calculate a formula for the minimum speed at which the car can drive around the track.

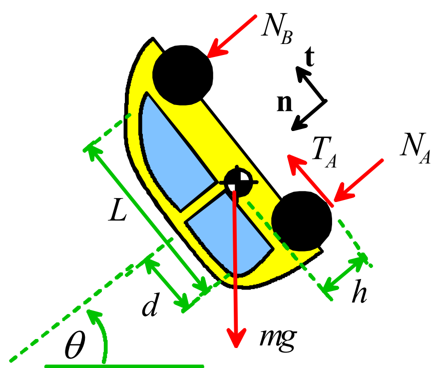


2.1 Assuming the car drives at constant speed V and its center of mass is a height h above the road, write down a formula for the acceleration of the vehicle in terms of V, the radius R of the track and h. Express your answer as components on the normal-tangential basis shown in the figure.

$$\mathbf{a} = \frac{V^2}{R-h}\mathbf{n}$$

[2 POINTS]

2.2 Draw a free body diagram showing the forces acting on the vehicle on the figure provided below. Assume that the car has rear wheel drive, and that the front wheels roll freely.



[2 POINTS]

2.3 Write down Newton's law of motion and the equation for rotational motion for the vehicle. Express your answer in normal-tangential components.

$$\mathbf{F} = (N_A + N_B + mg\sin\theta)\mathbf{n} + (T - mg\cos\theta)\mathbf{t} = m\frac{V^2}{(R-h)}\mathbf{n}$$

$$Th + N_B(L-d) - N_A d = 0$$

[2 POINTS]

2.4 Hence show that the reaction forces acting on the front wheel is

$$N_B = mg\left[\frac{V^2}{(R-h)\mathrm{g}}\frac{d}{L} - \left(\frac{d}{L}\sin\theta + \frac{h}{L}\cos\theta\right)\right]$$

The vector equation shows that

$$T = mg\cos\theta$$

$$N_A + N_B = m\frac{V^2}{(R-h)} - mg\sin\theta$$

Substituting for T in the moment balance equation

$$N_A d - N_B(L-d) = mg\cos\theta h$$

Multiplying the second equation by (L-d) and adding the third shows that

$$N_A L = m\frac{V^2}{R-h}(L-d) + mg\left[-(L-d)\sin\theta + h\cos\theta\right]$$

Multiplying the second equation by d and subtracting the third gives

$$LN_B = m\frac{V^2}{(R-h)}d - mg\left(d\sin\theta + h\cos\theta\right)$$

Hence

$$N_A = mg\left[\frac{V^2}{g(R-h)}(1-\frac{d}{L}) - (1-\frac{d}{L})\sin\theta + \frac{h}{L}\cos\theta\right]$$

$$N_B = mg\left[\frac{V^2}{(R-h)g}\frac{d}{L} - \left(\frac{d}{L}\sin\theta + \frac{h}{L}\cos\theta\right)\right]$$

[3 POINTS]

2.4 Hence, calculate a formula for the minimum speed required to ensure that the front wheels remain in contact with the track, in terms of h, L, g, d and R.

To remain in contact with the road the reactions must both be positive for all values of θ. For the front wheel

$$\frac{V^2}{(R-h)}d \geq g\left(h\cos\theta + d\sin\theta\right)$$

One can maximize the right hand side – differentiate wrt θ and set to zero – this shows that

$$-h\sin\theta + d\cos\theta = 0 \Rightarrow \theta = \tan^{-1}\frac{d}{h} \Rightarrow \cos\theta = \frac{h}{\sqrt{d^2+h^2}} \quad \sin\theta = \frac{d}{\sqrt{d^2+h^2}}$$

$$\Rightarrow \left(h\cos\theta + d\sin\theta\right)_{\max} = \sqrt{d^2+h^2}$$

,or (more quickly) note that we can write

$$g\left(h\cos\theta + d\sin\theta\right) = g\sqrt{h^2+d^2}\left(\frac{h}{\sqrt{h^2+d^2}}\cos\theta + \frac{d}{\sqrt{h^2+d^2}}\sin\theta\right) = g\sqrt{h^2+d^2}\left(\cos\alpha\cos\theta + \sin\alpha\sin\theta\right)$$

$$= g\sqrt{h^2+d^2}\cos(\theta-\alpha) \qquad \alpha = \tan^{-1}\frac{d}{h}$$

and the maximum value of $\cos(\theta-\alpha)$ is 1.

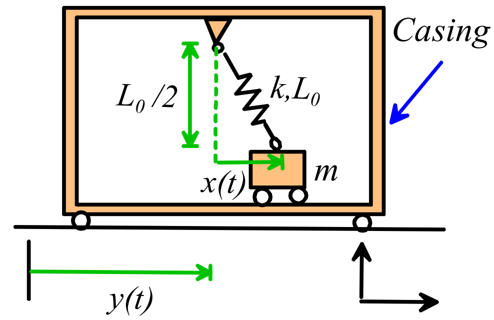
Therefore $N_B > 0 \Rightarrow V > \sqrt{(R-h)g\sqrt{1+\frac{h}{d^2}^2}}$

The same approach for N_A shows that

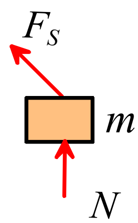
$$N_A > 0 \Rightarrow V > \sqrt{(R-h)g\sqrt{1+\frac{h^2}{(L-d)^2}}}$$

The critical velocity is given by the larger of the two results (only the front wheel calculation required for credit). **[4 POINTS]**

3. The figure shows a design for a micro-electro-mechanical shock sensor. It consists of a proof mass *m* inside a casing. The mass can move inside the casing, and is held in place by a spring with stiffness *k* and un-stretched length L_0. The casing moves horizontally with a given displacement *y(t)* relative to a fixed origin. The motion of the mass $x(t)$ relative to the casing is measured electrically.

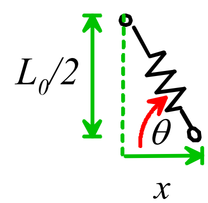


3.1Draw a free body diagram showing the forces acting on the mass. Gravity may be neglected



[2 POINTS]

3.2 Write down the (actual) length of the spring and a formula for $\cos\theta$ in terms of x and L_0



Pythagoras $L = \left(\sqrt{L_0^2/4 + x^2}\right)$ $\quad \cos\theta = \frac{x}{\sqrt{L_0^2/4+x^2}}$

[1 POINT]

3.2 Use Newton's laws of motion to show that *x(t)* satisfies the differential equation

$$m\frac{d^2x}{dt^2} + k\left(x - \frac{L_0 x}{\sqrt{L_0^2/4+x^2}}\right) = -m\frac{d^2y}{dt^2}$$

The acceleration of the mass is $\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2}$. Newton's law in the **I** direction gives

$$m\left(\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2}\right) = -F_s\frac{x}{\sqrt{L_0^2/4+x^2}}$$

The spring force law gives $F_s = k\left(\sqrt{L_0^2/4+x^2} - L_0\right)$

Thus

$$m\frac{d^2x}{dt^2} + k\left(x - \frac{L_0 x}{\sqrt{L_0^2/4+x^2}}\right) = -m\frac{d^2y}{dt^2}$$

[2 POINTS]

3.3 Re-arrange the equation into a form that MATLAB could solve

We must re-arrange the second order ODE into two first-order ODEs. The standard approach gives

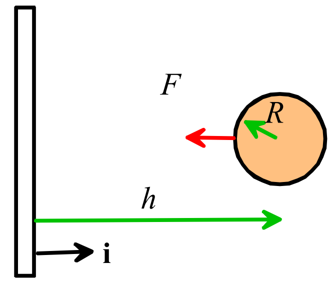
$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} v \\ -\frac{k}{m}\left(x - \frac{L_0 x}{\sqrt{L_0^2 / 4 + x^2}} \right) - \frac{d^2 y}{dt^2} \end{bmatrix}$$

[2 POINTS]

4. A uniformly charged spherical particle of laser-printer toner experiences a force of attraction to a conductive surface given by

$$F = \frac{K}{h^2}$$

where K is a constant and h is the distance of the center of the particle from the surface.



4.1 Show that the potential energy of the particle can be expressed as

$$V = -\frac{K}{h} + C$$

Where C is a constant.

By definition $V = -\int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{F} \cdot d\mathbf{r} = -\int_{h_A}^{h} -\frac{K}{x^2}\mathbf{i} \cdot (dx\mathbf{i}) = -K\left(\frac{1}{h} - \frac{1}{h_A} \right)$

and we can choose $K / h_A = c$

[2 POINTS]

4.2 A charged toner particle with mass m is launched with speed v_0 towards a conducting surface from a point infinitely far from the surface. Find a formula for the speed v_1 of the particle just before it impacts the surface.

Use energy conservation: $\frac{1}{2}mv_0^2 + C = \frac{1}{2}mv_1^2 - \frac{K}{R} + C \Rightarrow v_1 = \sqrt{\frac{2K}{mR} + v_0^2}$

[2 POINTS]

4.3 Assume that the impact between the particle and the surface has restitution coefficient e. Calculate the velocity vector of the particle just after impact. The surface remains stationary before and after imact.

$$v_2\mathbf{i} = -e(-v_1\mathbf{i}) \Rightarrow \mathbf{v}_2 = ev_1\mathbf{i}$$

[1 POINT]

4.4 Hence, find a formula for the maximum distance from the surface that the particle will reach after the rebound, in terms of K, R, e, m and v_0.

At the point of maximum distance the velocity is zero. Energy is conserved after rebound, so

$$\frac{1}{2}mv_2^2 - \frac{K}{R} + C = -\frac{K}{h} + C$$

$$e^2\left(\frac{K}{R} + \frac{1}{2}mv_0^2 \right) - \frac{K}{R} = -\frac{K}{h} \Rightarrow \frac{K}{h} = \frac{K}{R}(1 - e^2) - e^2 \frac{1}{2}mv_0^2$$

$$\Rightarrow h = \frac{K}{K(1 - e^2) / R - e^2 mv_0^2 / 2}$$

[2 POINTS]

4.5 Show that the particle will eventually attach to the surface if the initial speed v_0 is less than a critical value, and find a formula for the critical speed in terms of *e*, *m, K* and *R*.

The particle will escape from the surface if $h \rightarrow \infty$. For a finite positive *h* it will be attracted back to the surface and lose energy at each impact – eventually it will come to rest attached to the surface.

The critical velocity for $h \rightarrow \infty$ satisfies

$$K(1-e^2)/R - e^2 m v_0^2 / 2 = 0$$

$$\Rightarrow v_0 = \sqrt{\frac{2K(1-e^2)}{mRe^2}}$$

[3 POINTS]