School of Engineering
Brown University

EN40: Dynamics and Vibrations

Midterm Examination
Thursday April 2 2015

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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1 (5 points) _____________

2. (10 points) _____________

3. (5 points) _____________

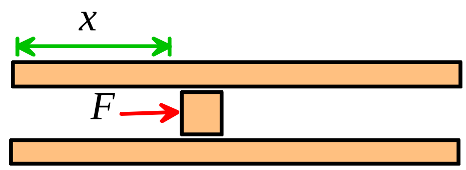
4. (10 points) _____________

TOTAL (30 points) ____________

1. The projectile in a gas gun is subjected to a propulsive force

$$F = F_0\left(1-\frac{v}{3c}\right)^5$$

where F_0, c are constants and v is the projectile's speed. The projectile has mass m. It starts at rest at time $t=0$ at position $x=0$.



1.1 Use Newton's law to determine the acceleration of the projectile and hence determine an expression for its speed as a function of time and other parameters. Air resistance and friction may be neglected.

$$F = ma \Rightarrow m\frac{dv}{dt} = F_0\left(1-\frac{v}{3c}\right)^5$$

$$\Rightarrow \int_0^v \left(1-\frac{v}{3c}\right)^{-5} dv = \frac{F_0}{m}\int_0^t dt$$

$$\Rightarrow \frac{3c}{4}\left(1-\frac{v}{3c}\right)^{-4} - \frac{3c}{4} = \frac{F_0}{m}t$$

$$\Rightarrow v = 3c\left[1-\frac{1}{\left(1+4F_0t/3cm\right)^{1/4}}\right]$$

[3 POINTS]

1.2 Find a formula for the distance traveled by the projectile as a function of time.

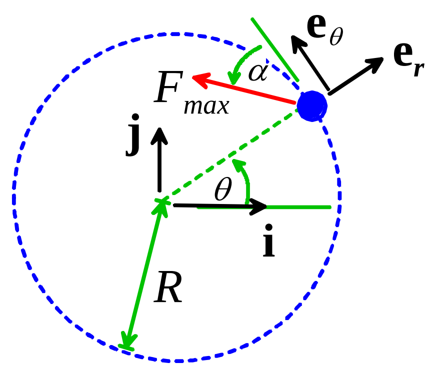
$$\frac{dx}{dt} = 3c\left[1-\frac{1}{\left(1+4F_0t/3cm\right)^{1/4}}\right]$$

$$\Rightarrow x = \int_0^t 3c\left[1-\frac{1}{\left(1+4F_0t/3cm\right)^{1/4}}\right]$$

$$\Rightarrow x = 3ct - \frac{3mc^2}{F_0}\left[\left(1+4F_0t/3cm\right)^{3/4}-1\right]$$

[2 POINTS]

2. A 'prey' particle with mass m is subjected to a propulsive force with magnitude $F_{\max}$ that is applied at a constant angle α to its direction of motion. It also experiences a drag force (not shown in the figure) with magnitude cV (where V is the magnitude of the velocity and c is a constant) that acts opposite to its direction of motion. The random force is zero. As a result, it travels at constant speed V around a circular path with radius R.



2.1 Write down expressions for the propulsive force vector and drag force vector acting on the particle, in terms of $F_{\max}$, α, c and V, expressing your answer in polar coordinates $\{\mathbf{e}_r, \mathbf{e}_\theta\}$.

$$\mathbf{F}_P = F_{\max}\cos\alpha\,\mathbf{e}_\theta - F_{\max}\sin\alpha\,\mathbf{e}_r$$
$$\mathbf{F}_D = -cV\mathbf{e}_\theta$$

[2 POINTS]

2.2 Write down an expression for the acceleration vector in terms of V and R, and hence use Newton's law to show that

$$V = \frac{F_{\max}}{c}\cos\alpha \qquad R = \frac{mF_{\max}}{c^2}\frac{\cos^2\alpha}{\sin\alpha}$$

The circular motion formula gives

$$\mathbf{a} = -\frac{V^2}{R}\mathbf{e}_r$$

$$\mathbf{F} = m\mathbf{a} \Rightarrow -F_{\max}\sin\alpha\,\mathbf{e}_r + (F_{\max}\cos\alpha - cV)\mathbf{e}_\theta = -m\frac{V^2}{R}\mathbf{e}_r$$

The hoop direction equation shows that

$$V = \frac{F_{\max}}{c}\cos\alpha$$

and the radial equation then gives

$$R = \frac{mV^2}{F_{\max}\sin\alpha} = \frac{mF_{\max}}{c^2}\frac{\cos^2\alpha}{\sin\alpha}$$

[3 POINTS]

2.3 Find a formula for the time required for the particle to complete a full circle.

$$T = 2\pi R/V = 2\pi\frac{m}{c}\frac{\cos\alpha}{\sin\alpha}$$

[1 POINT]

2.4 In the MATLAB code controlling the prey, the propulsive force is calculated from the formula $\mathbf{F} = F_{\max}\cos\omega t\mathbf{i} + F_{\max}\sin\omega t\mathbf{j}$, where ω is a constant. Use the solutions to 2.2 and 2.3 to find a formula relating the radius of the path R to ω, $F_{\max}$, m and c. The formula $\cos\theta = 1/\sqrt{1+\tan^2\theta}$ might be helpful.

The applied force has a period $T = 2\pi/\omega$. Setting this equal to the period calculated in 2.3 shows that

$$\tan\alpha = m\omega/c$$

$$\Rightarrow \cos\alpha = 1/\sqrt{1+(m\omega/c)^2}$$

Hence the radius of the path follows as

$$R = \frac{mF_{\max}}{c^2}\frac{\cos^2\alpha}{\sin\alpha} = \frac{F_{\max}}{\omega c\sqrt{1+(m\omega/c)^2}} = \frac{F_{\max}}{\omega\sqrt{c^2+m^2\omega^2}}$$

[2 POINTS]

2.5 The prey starts with a total energy supply E. Find a formula for the maximum time that the particle can continue traveling around the circular path without exhausting the energy supply, in terms of $E, F_{\max}, m, \omega, c$.

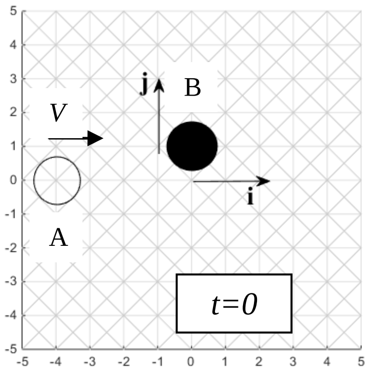
The power expenditure can be calculated as either $P = F_{\max}\cos\alpha V$ or $P = cV^2$ which gives

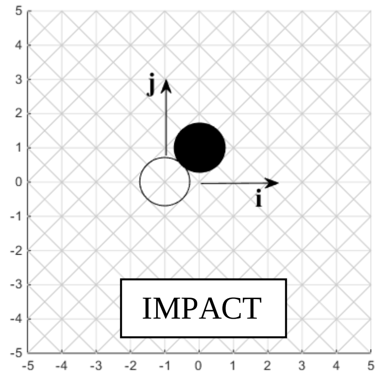
$$P = \frac{F_{\max}^2}{c}\cos^2\alpha = \frac{F_{\max}^2}{c\left(1+(m\omega/c)^2\right)}$$

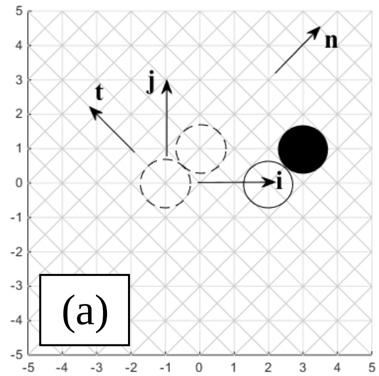
The time to exhaust the energy is therefore $t = E/P = \dfrac{Ec}{F_{\max}^2}\left(1+\dfrac{m^2\omega^2}{c^2}\right)$

[2 POINTS]

3. Two spheres with identical mass and restitution coefficient *e=0* have initial positions shown in the figure below. Before impact sphere B is stationary and sphere *A* has velocity *V***i.** The collision is frictionless. By answering the true/false questions below, identify which of the figures (a-d) shows the correct position of the spheres after collision.







CIRCLE ONE RESPONSE TO EACH STATEMENT BELOW

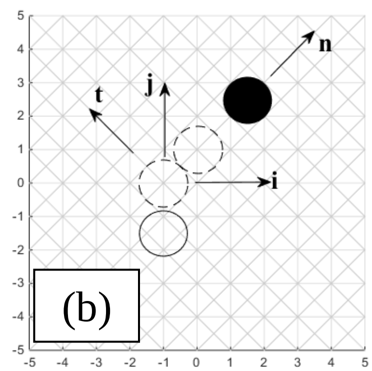
(a) Total Momentum is conserved in the **j** direction **T** F
Momentum of B is conserved in the **t** direction T **F**
The restitution formula is satisfied in the **n** direction **T** F

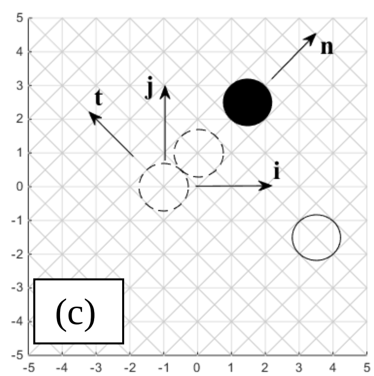
(b) Total Momentum is conserved in the **j** direction **T** F
Momentum of B is conserved in the **t** direction **T** F
The restitution formula is satisfied in the **n** direction T **F**

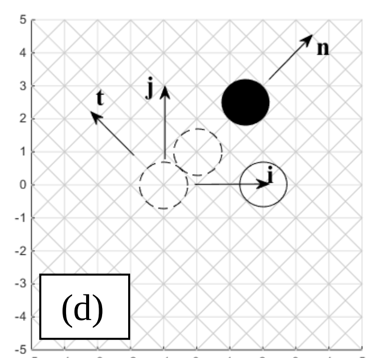
(c) Total Momentum is conserved in the **j** direction **T** F
Momentum of B is conserved in the **t** direction **T** F
The restitution formula is satisfied in the **n** direction **T** F

(d) Total Momentum is conserved in the **j** direction T **F**
Momentum of B is conserved in the **t** direction **T** F
The restitution formula is satisfied in the **n** direction **T** F

Correct figure: a b **c** d

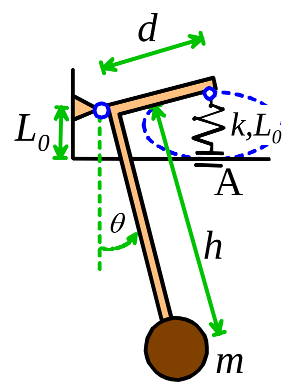






Total momentum in the **j** direction before impact is zero, and must therefore be zero after impact. That means A and B must have equal and opposite vertical velocities (and must travel equal and opposite distances).
Momentum of B in the **t** direction is zero before impact and must be zero after impact. B must therefore travel parallel to **n**
Since the restitution coefficient is zero, the restitution formula in the **n** direction requires that both spheres have the same velocity in the **n** direction after impact. This means that both spheres must travel the same distance parallel to **n** after impact

4. The figure shows a schematic diagram of a pendulum apparatus designed to measure the stiffness of an intervertebral disk. The disk is idealized as a spring with stiffness k and unstretched length L_0 (the slider at A allows the spring to remain vertical at all times). The goal of this problem is to find the relationship between k and the measured period of oscillation of the pendulum.



4.1 Write down a formula for the speed of the mass m in terms of h and $d\theta / dt$

Circular motion formula: $V = h\left(\frac{d\theta}{dt}\right)$

[1 POINT]

4.2 Hence, write down the total potential and kinetic energy of the system, in terms of k, d, h, m, g and θ and its time derivatives.

$$E = \frac{1}{2}m\left(h\frac{d\theta}{dt}\right)^2 + \frac{1}{2}kd^2\sin^2\theta - mgh\cos\theta$$

[2 POINTS]

4.3 Hence, show that θ satisfies the equation of motion

$$mh^2\frac{d^2\theta}{dt^2} + kd^2\sin\theta\cos\theta + mgh\sin\theta = 0$$

$$\frac{dE}{dt} = mh^2\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} + kd^2\sin\theta\cos\theta\frac{d\theta}{dt} + mgh\sin\theta\frac{d\theta}{dt} = 0$$

$$\Rightarrow mh^2\frac{d^2\theta}{dt^2} + kd^2\sin\theta\cos\theta + mgh\sin\theta = 0$$

[2 POINTS]

4.4 Linearize the equation of motion for small θ and hence find a formula for the natural frequency of vibration of the pendulum, in terms of k, d, h, m, g.

Recall $\sin\theta \approx \theta \quad \cos\theta \approx 1$

$$mh^2\frac{d^2\theta}{dt^2} + \left(kd^2 + mgh\right)\theta = 0$$

$$\Rightarrow \frac{mh^2}{\left(kd^2 + mgh\right)}\frac{d^2\theta}{dt^2} + \theta = 0$$

The natural frequency follows as $\omega_n = \sqrt{\frac{kd^2}{mh^2} + \frac{g}{h}}$

[3 POINTS]

4.5 Rearrange the equation in 4.3 into a form that MATLAB could solve using the ode45 function.

The equation must be re-written as two first-order equations

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ -kd^2 \sin\theta \cos\theta / (mh^2) - (g/h)\sin\theta\end{bmatrix}$$

[2 POINTS]