School of Engineering
Brown University

EN40: Dynamics and Vibrations

Midterm Examination
Tuesday March 8 2016

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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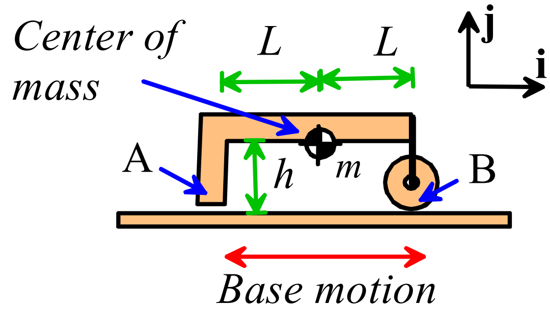
1 (19 points) _____________

2. (11 points) _____________

3. (10 points) _____________

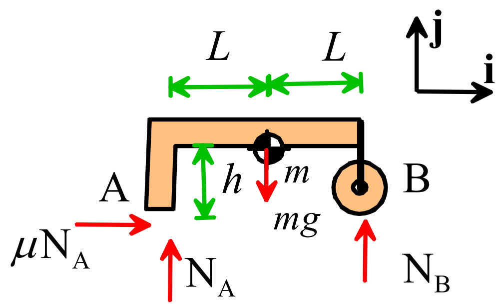
TOTAL (38 points)_____________



1. The figure shows an 'inertial crawler' that will spontaneously translate to the right over a vibrating surface. Assume that

- The device and surface are both at rest at time $t=0$.
- For the time interval $0<t<T$ (where T is a constant) the surface has velocity
$$\mathbf{v}=\begin{cases}+V_0\mathbf{i} & 0<t<T/2\\ -V_0\mathbf{i} & T/2<t<T\end{cases}$$
- V_0 is sufficiently large that slip occurs at A throughout the interval $0<t<T$.
- The contact at A has a friction coefficient μ.
- The wheel at B is massless and rolls freely (**so the contact at B is subjected only to a normal force**).
- The center of mass of the device is midway between A and B and a height h above the surface.

1. Draw a free body diagram showing the forces acting on the crawler during the period $0<t<T/2$ on the figure provided below. Note that during this phase of the motion the surface moves to the right, and is moving faster than the crawler.



[3 POINTS]

2. Write down Newton's law and the equation of rotational motion for $0<t<T/2$

$$\mu N_A\mathbf{i}+(N_A+N_B-mg)\mathbf{j}=ma\mathbf{i}$$
$$-\left(N_AL-\mu N_Ah-N_BL\right)\mathbf{k}=0$$

[3 POINTS]

3. Hence, show that the acceleration of the crawler for $0<t<T/2$ is given by

$$a=\frac{\mu g}{2-\mu(h/L)}\mathbf{i}$$

$$\mu N_A\mathbf{i}+(N_A+N_B-mg)\mathbf{j}=ma\mathbf{i}$$
$$N_AL-\mu N_Ah-N_BL=0$$
$$N_A+N_B-mg \qquad \Rightarrow N_A \quad + \quad N_B=mg$$
$$N_AL-\mu N_Ah-N_BL=0\Rightarrow \quad N_A\left(1-\mu h/L\right)-N_B=0$$
$$Add: \qquad N_A\left(2-\mu h/L\right)=mg$$

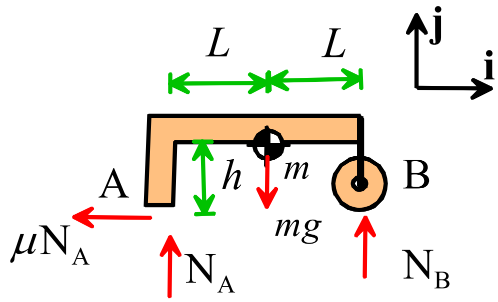
$$\mu N_A=ma\Rightarrow a=\frac{\mu g}{\left(2-\mu h/L\right)}$$

[3 POINTS]

4. Repeat steps 1-3 for the period $T/2 < t < T$ (note that during this phase of the motion the surface moves to the left) to show that during this time interval

$$a = \frac{-\mu g}{2+\mu(h/L)}\mathbf{i}$$

New FBD



New EOM

$$-\mu N_A\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = ma\mathbf{i}$$
$$N_A L + \mu N_A h - N_B L = 0$$

New solution

$$\begin{aligned}
&N_A + N_B - mg && \Rightarrow N_A + N_B = mg \\
&N_A L + \mu N_A h - N_B L = 0 \Rightarrow && N_A\left(1+\mu h/L\right) - N_B = 0 \\
&Add: && N_A\left(2+\mu h/L\right) = mg
\end{aligned}$$

$$-\mu N_A = ma \Rightarrow a = \frac{-\mu g}{\left(2+\mu h/L\right)}$$

[4 POINTS]

5. Use the results of (3) and (4) to find a formula for the distance that the crawler moves during the time period 0<*t*<T. Assume that the crawler is at rest at time *t=0.*

The acceleration is constant during both phases of motion so we can use the straight-line motion formulas:

$$0 < t < T/2: \quad v = at = \frac{\mu g t}{2-\mu(h/L)} \qquad x = \frac{1}{2}at^2 = \frac{1}{2}\frac{\mu g t^2}{2-\mu(h/L)}$$

$$At \;\; t = T/2 \qquad v = \frac{\mu g T/2}{2-\mu(h/L)} \qquad x = \frac{1}{2}\frac{\mu g T^2/4}{2-\mu(h/L)}$$

$$T/2 < t < T: \quad v = v_0 + a(t - t_0) = \frac{\mu g T/2}{2 - \mu(h/L)} - \frac{\mu g(t - T/2)}{2 + \mu(h/L)}$$

$$x = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 = \frac{1}{2}\frac{\mu g T^2/4}{2 - \mu(h/L)} + \frac{\mu g T/2}{2 - \mu(h/L)}(t - T/2) + \frac{1}{2}\frac{-\mu g(t - T/2)^2}{2 + \mu(h/L)}$$

$$At \;\; t = T \qquad x = \frac{1}{2}\frac{\mu g T^2/4}{2 - \mu(h/L)} + \frac{\mu g T^2/4}{2 - \mu(h/L)} + \frac{1}{2}\frac{-\mu g T^2/4}{2 + \mu(h/L)}$$

$$= \frac{\mu g T^2}{8}\left(\frac{3}{2 - \mu(h/L)} - \frac{1}{2 + \mu(h/L)}\right) = \frac{\mu g T^2(4 + 4\mu h/L)}{8(4 - (\mu h/L)^2)} = \frac{\mu g T^2(1 + \mu h/L)}{2(4 - (\mu h/L)^2)}$$

[4 POINTS]

6. Find a formula for the minimum value of V_0 for slip to occur at the contact point A between the crawler and the surface in the time interval $0<t<T$.

   The surface must be moving faster than the crawler throughout the time interval. This requires

$$V_0 > \frac{\mu g T/2}{2 - \mu(h/L)}$$

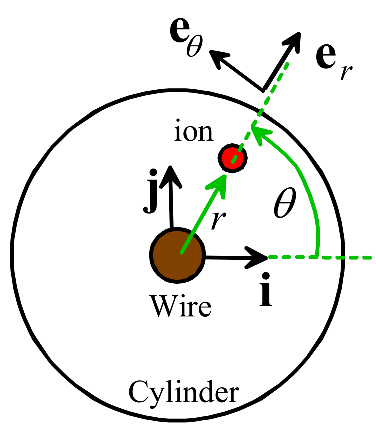
[2 POINTS]

2. The figure shows a charged particle with mass m in a static 'Kingdon trap' that uses an electric field to confine the particle within a cylinder. The electric field subjects the particle to a radial force

$$\mathbf{F} = -\frac{F_0}{r}\mathbf{e}_r$$

where F_0 is a constant (which depends on the electric field in the cylinder).

At time $t=0$ the particle is located at position $\theta = 0, r = R_0$ and has velocity vector $\mathbf{v} = V_0 \mathbf{e}_\theta$



2.1 Write down the formula for the acceleration of the particle in polar coordinates, in terms of time derivatives of r, θ . (Do not assume circular motion).

This is just the polar coordinate acceleration formula $\mathbf{a} = \left\{ \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right\}\mathbf{e}_r + \left\{ r\frac{d^2\theta}{dt^2} - 2\frac{dr}{dt}\frac{d\theta}{dt} \right\}\mathbf{e}_\theta$

[2 POINTS]

2.2 Hence, write down Newton's law **F**=m**a** for the particle, using polar coordinates.

$$-\frac{F_0}{r}\mathbf{e}_r = m\left\{ \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right\}\mathbf{e}_r + m\left\{ r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \right\}\mathbf{e}_\theta$$

[1 POINT]

2.3 Write down the angular momentum vector of the particle at time $t=0$, in terms of V_0 and R_0

From the angular momentum formula $\mathbf{h} = \mathbf{r} \times m\mathbf{v} = R_0 \mathbf{e}_r \times m V_o \mathbf{e}_\theta = m R_0 V_0 \mathbf{k}$

[1 POINT]

2.4 Use angular momentum, or otherwise, show that θ and r are related by

$$\frac{d\theta}{dt} = \frac{V_0 R_0}{r^2}$$

This is a central force problem so angular momentum is conserved. At a later time $\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta$,

and $\mathbf{r} \times m\mathbf{v} = r\mathbf{e}_r \times m\left(\frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta\right) = r^2 m \frac{d\theta}{dt}\mathbf{k}$

Angular momentum conservation gives $m R_0 V_0 = r^2 m \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{V_0 R_0}{r^2}$

You can also get the same result from the $\mathbf{e}_\theta$ component of **F**=m**a** (part 2.1) which shows

$$r\frac{d^2\theta}{dt^2}+2\frac{dr}{dt}\frac{d\theta}{dt}=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0$$

This can be integrated to get

$$\left(r^2\frac{d\theta}{dt}\right)=const$$

And we can find the constant is R_0V_0 from the initial conditions.

[2 POINTS]

2.5 Use the answers to 2.2 and 2.4 to show that the coordinate *r* satisfies the differential equation

$$\frac{d^2r}{dt^2}-\frac{(V_0R_0)^2}{r^3}+\frac{F_0}{mr}=0$$

Part 2.2 shows that

$$m\left\{\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right\}=-\frac{F_0}{r}$$

Substituting $\frac{d\theta}{dt}=\frac{V_0R_0}{r^2}$ from part 2.4 gives

$$\frac{d^2r}{dt^2}-r\left(\frac{V_0R_0}{r^2}\right)^2+\frac{F_0}{mr}=0$$

This gives the required answer.

[3 POINTS]

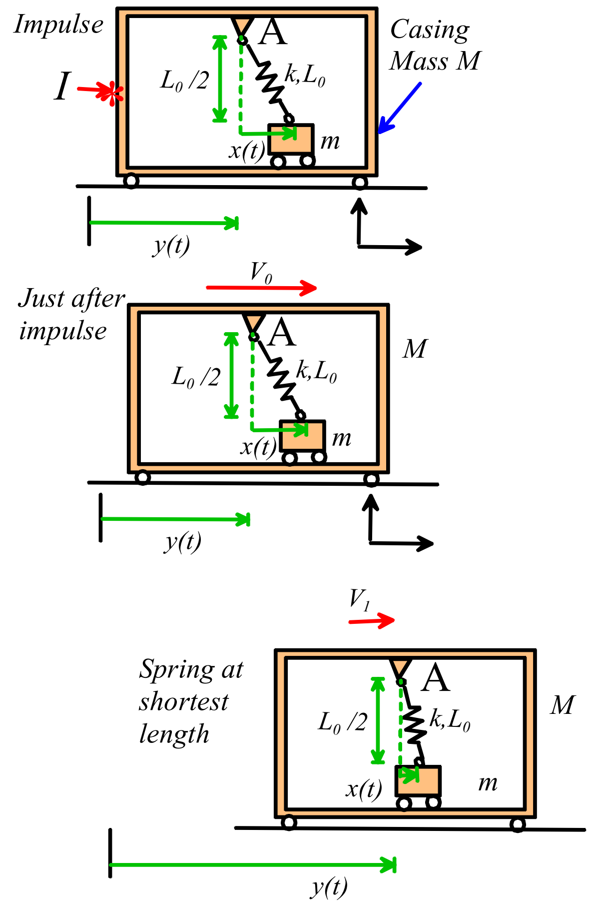
2.6 Rearrange the equation of motion for *r* in part 3.3 into a form that MATLAB can solve.

We need to transform the second order equation into two first order equations. Let $v=\frac{dr}{dt}$, then

$$\frac{d}{dt}\begin{bmatrix} r \\ v \end{bmatrix}=\begin{bmatrix} v \\ \frac{(V_0R_0)^2}{r^3}-\frac{F_0}{mr} \end{bmatrix}$$

[2 POINTS]

3. The figure shows a device that is intended to detect an impulse. If the casing is subjected to an impulse that exceeds a critical magnitude, the mass will flip from its initial position to the right of the pivot to a new stable position to the left of the pivot at A. The goal of this problem is to calculate the critical value of impulse for which this will occur.



3.1 At time $t=0$ the system is at rest and the spring is un-stretched. The casing is then subjected to a horizontal impulse I. Write down a formula for the speed of the casing just after the impulse. Note that the spring exerts no force on either the casing or the mass m during the impulse.

$$MV_0 = I \Rightarrow V_0 = I / M$$

[1 POINT]

3.2 Find expressions for the total linear momentum and total kinetic energy of the system just after the impulse, in terms of I and the mass M of the casing.

$$T = \frac{1}{2}MV_0^2 = \frac{1}{2M}I^2$$
$$\mathbf{p} = MV_0\mathbf{i} = I\mathbf{i}$$

[2 POINTS]

3.3 Consider the system at the instant when the spring reaches its shortest length (assume $x>0$). Using energy and/or momentum conservation show that at this instant

$$x = \sqrt{\left(L_0 - I\sqrt{\frac{m}{kM(M+m)}}\right)^2 - \frac{L_0^2}{4}}$$

(You can find the spring length using Pythagoras' theorem)

- At this instant both the casing and the proof mass move with the same (unknown) speed v_1 .
- Momentum conservation requires that $(m+M)v_1 = I$
- Energy conservation requires that $\frac{1}{2M}I^2 = \frac{1}{2}(M+m)v_1^2 + \frac{1}{2}k\left\{L_0 - \sqrt{x^2 + L_0^2/4}\right\}^2$
  Here the first term on the right is the KE, the second is the PE of the spring. We used $L_0 - \sqrt{x^2 + L_0^2/4}$ instead of $\sqrt{x^2 + L_0^2/4} - L_0$ for convenience (it makes the quantity inside the $\{\ \}$ positive)
- Eliminate v_1 :

$$\frac{1}{2}\left(\frac{1}{M}-\frac{1}{M+m}\right)I^2=\frac{1}{2}k\left\{L_0-\sqrt{x^2+L_0^2/4}\right\}^2$$

$$\Rightarrow x=\sqrt{\left(L_0-I\sqrt{\frac{m}{kM(M+m)}}\right)^2-\frac{L_0^2}{4}}$$

[5 POINTS]

3.4 Hence, find a formula for the critical value of I that will flip the mass past $x=0$.

- Set $x=0$ in the preceding problem and solve for I: $I=\frac{L_0}{2}\sqrt{kM\left(\frac{M}{m}+1\right)}$

[2 POINTS]