School of Engineering
Brown University

EN40: Dynamics and Vibrations

Midterm Examination
Thursday March 9 2017

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

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1 (10 points) _____________

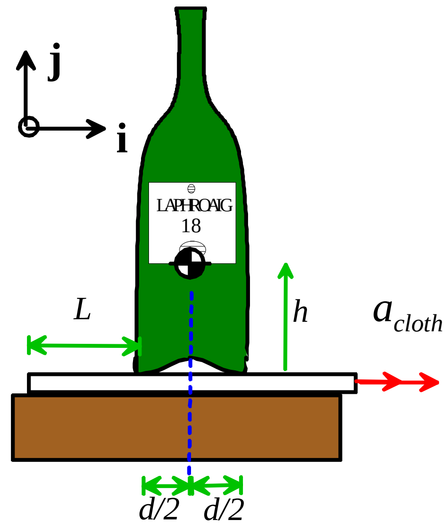
2. (12 points) _____________

3. (3 points) _____________

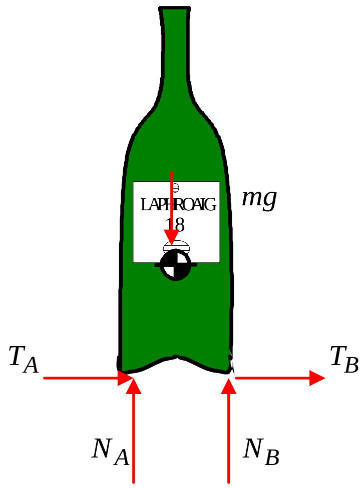
4. (10 points) _____________

TOTAL (35 points) ____________

1. A bottle with mass m rests on a table-cloth. The contact between them has friction coefficient μ . At time $t=0$ the object and cloth are both stationary. For time $t>0$, the cloth is pulled with constant acceleration $a_{cloth} = 5\mu g$ the right. Note that since $a_{cloth} > \mu g$, slip must occur at the contact just after $t=0$.



1.1 Draw a free body diagram showing the forces acting on the object on the figure below.



[3 POINTS]

1.2 Find a formula for the horizontal acceleration of the bottle. Assume that the bottle does not tip over.

F=m**a** gives $(T_A + T_B)\mathbf{i} + (N_A + N_B - mg)\mathbf{j} = ma_x\mathbf{i}$

The friction law (since there is slip) gives $T_A = \mu N_A \quad T_B = \mu N_B$

Hence $N_A + N_B = mg, \qquad a_x = \frac{T_A + T_B}{m} = \mu \frac{N_A + N_B}{m} = \mu g$

[2 POINTS]

1.3 Find a formula for the horizontal distance moved by the bottle as a function of time.

The straight line motion formula gives $x = \frac{1}{2}\mu g t^2$

[1 POINT]

1.4 Find a formula for the distance moved by the cloth as a function of time

The straight line motion formula gives $x_{cloth} = \frac{5}{2}\mu g t^2$

[1 POINTS]

1.5 At time *t=0* the bottle is a distance *L* from the edge of the cloth. Calculate the distance that the bottle has moved in the **i** direction at the instant that the edge of the cloth reaches the edge of the bottle (*L=0* at this instant). Express your answer in terms of *L* and any other parameters you think are relevant.

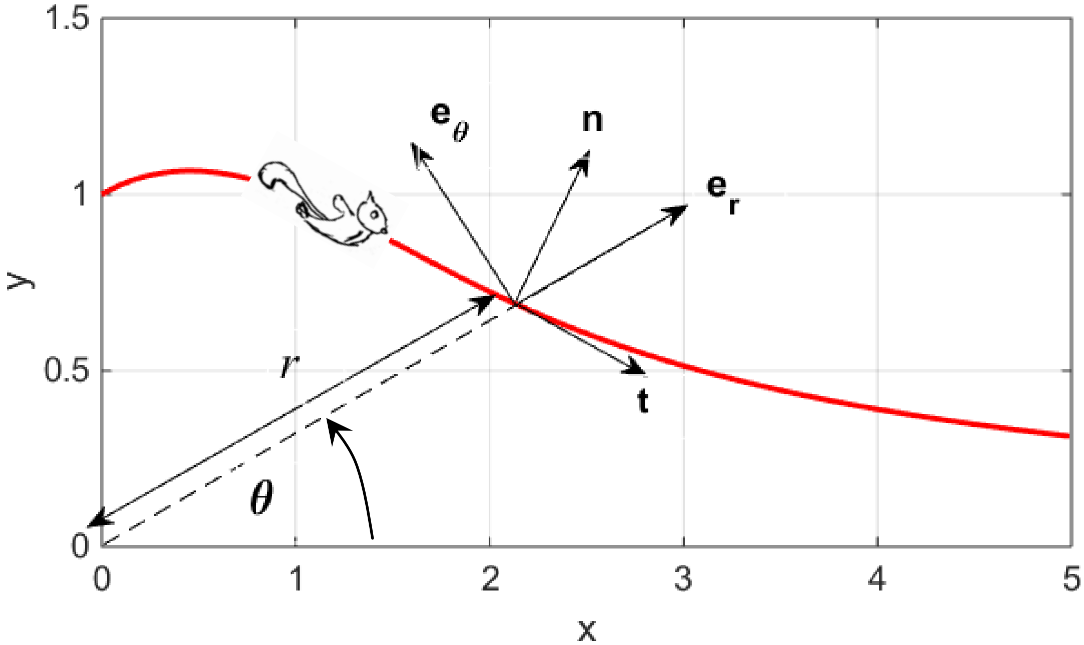
The cloth must move a distance *L* further than the bottle, so

$$x_{cloth} - x = \frac{4}{2}\mu g t^2 = L \Rightarrow t = \sqrt{\frac{L}{2\mu g}}$$

The bottle therefore moves a distance

$$x = \frac{1}{2}\mu g \frac{L}{2\mu g} = \frac{1}{4}L$$

[3 POINTS]



2. The figure shows the trajectory of a flying squirrel, which is measured (in polar coordinates) to be

$$r = (1+t) \text{ (meters)} \qquad \theta = \frac{\pi}{2(1+t)^2} \quad (\textit{radians})$$

(Note that $\theta = \pi / 2$ at time t=*0*, and decreases as t increases).

2.1 Find a formula for the velocity vector (in polar coordinates $\{\mathbf{e}_r, \mathbf{e}_\theta\}$) terms of time

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta = \mathbf{e}_r - \frac{\pi}{(1+t)^2}\mathbf{e}_\theta$$

[1 POINT]

2.2 Find a formula for the acceleration vector in polar coordinates $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ as a function of time

$$\mathbf{a} = \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\mathbf{e}_r + \left\{r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right\}\mathbf{e}_\theta$$

$$= -\frac{\pi^2}{(1+t)^5}\mathbf{e}_r + \left\{(1+t).\frac{2.3.\pi}{2(1+t)^4} - 2.\frac{\pi}{(1+t)^3}\right\}\mathbf{e}_\theta$$

$$= -\frac{\pi^2}{(1+t)^5}\mathbf{e}_r + \frac{\pi}{(1+t)^3}\mathbf{e}_\theta = \frac{\pi}{(1+t)^5}\left\{-\pi\mathbf{e}_r + (1+t)^2\mathbf{e}_\theta\right\}$$

[2 POINTS]

2.3 Find unit vectors normal and tangent to the path of the squirrel **AT TIME t=0**, in the $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ basis. Choose the sign of **n** so that the **n** has a positive $\mathbf{e}_r$ component.

The tangent vector is parallel to **v**, and the normal vector can be found as $\mathbf{k} \times \mathbf{t}$, which gives

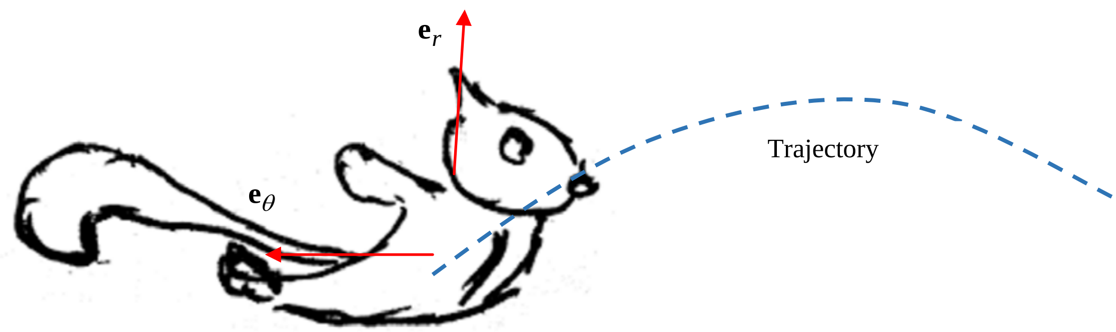
$$\mathbf{t} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{1+\pi^2}}\left(\mathbf{e}_r - \pi \mathbf{e}_\theta\right)$$

$$\mathbf{n} = \pm \mathbf{k} \times \mathbf{t} = \pm \frac{1}{\sqrt{1+\pi^2}}\left(\pi \mathbf{e}_r + \mathbf{e}_\theta\right)$$

Take the positive sign to give a positive $\mathbf{e}_r$ component.

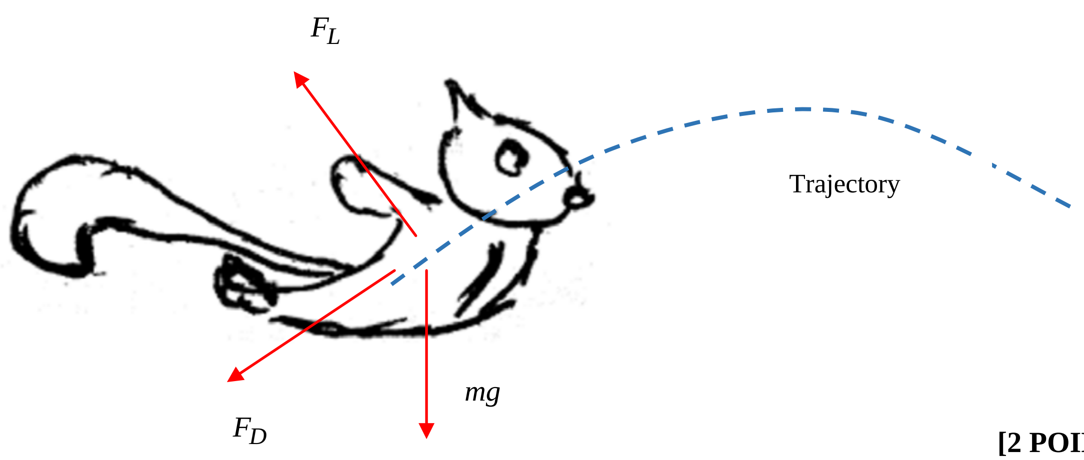
[2 POINTS]

2.4 Draw the unit vectors $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ **AT TIME t=0** on the figure provided below



[1 POINT]

2.5 Draw a free body diagram showing the forces acting on the squirrel **at time *t=0*** on the figure provided. Include lift and drag forces, and any other forces you think are relevant.



[2 POINTS]

2.6 If the squirrel has mass 100 grams, calculate the magnitude of the lift and drag force acting on the squirrel **at time *t=0*** (you can use the approximation $g \approx 10 m/s^2$ and leave your answer in terms of π - there is no need to get a number for the answer)

Newton's law **F**=m**a** gives

$$-F_D \mathbf{t} + F_L \mathbf{n} - mg\mathbf{j} = m\mathbf{a}$$

(we use the positive sign for **n** in . We can solve for F_D, F_L by taking the dot product with **t,n** and recalling $\mathbf{t}\cdot\mathbf{t} = \mathbf{n}\cdot\mathbf{n} = 1 \quad \mathbf{n}\cdot\mathbf{t} = 0$

$$\begin{aligned} F_D &= -m\mathbf{a}\cdot\mathbf{t} - mg\mathbf{j}\cdot\mathbf{t} \\ F_L &= m\mathbf{a}\cdot\mathbf{n} + mg\mathbf{j}\cdot\mathbf{n} \end{aligned}$$

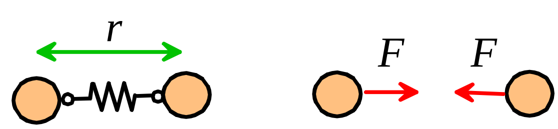
Note that at time *t=0* $\mathbf{j} = \mathbf{e}_r$, and we can substitute for the other vectors from previous parts of the problem

$$\begin{aligned} F_D &= -m\pi(-\pi\mathbf{e}_r + \mathbf{e}_\theta)\cdot\frac{1}{\sqrt{1+\pi^2}}(\mathbf{e}_r - \pi\mathbf{e}_\theta) - mg\mathbf{e}_r \cdot \frac{1}{\sqrt{1+\pi^2}}(\mathbf{e}_r - \pi\mathbf{e}_\theta) = m\frac{2\pi^2 - 10}{\sqrt{1+\pi^2}} \\ F_L &= m\pi(-\pi\mathbf{e}_r + \mathbf{e}_\theta)\cdot\frac{1}{\sqrt{1+\pi^2}}(\pi\mathbf{e}_r + \mathbf{e}_\theta) + mg\mathbf{e}_r \cdot \frac{1}{\sqrt{1+\pi^2}}(\pi\mathbf{e}_r + \mathbf{e}_\theta) = m\frac{\pi(11-\pi^2)}{\sqrt{1+\pi^2}} \end{aligned}$$

[4 POINTS]

3. The attractive force between two atoms in a diatomic molecule is related to the distance *r* between them by

$$F = 8F_0\left(\left(\frac{a}{r}\right)^5 - \left(\frac{a}{r}\right)^9\right)$$



where F_0 and *a* are constants. (**NB:** the spring in the figure is used to indicate the atomic bond – it does not obey the usual force-v-length relation for a linear spring)

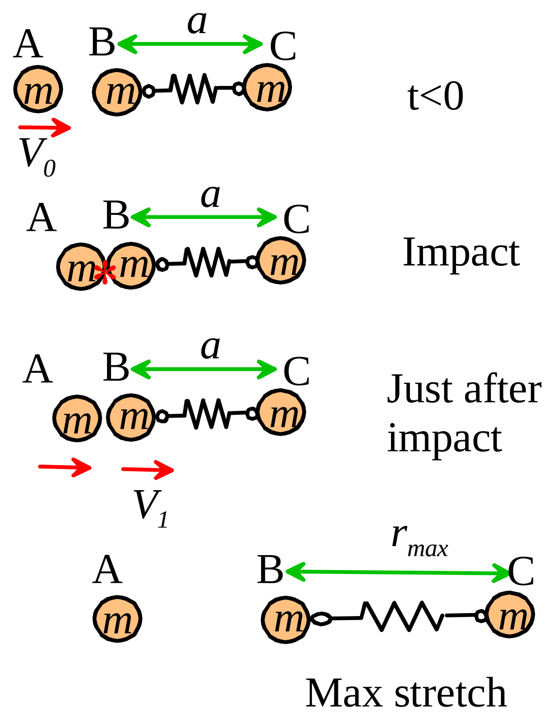
Show that the potential energy of the interatomic bond is (to within an arbitrary constant)

$$V(r) = F_0 a\left(1 - 2\left(\frac{a}{r}\right)^4 + \left(\frac{a}{r}\right)^8\right)$$

By definition $V = -\int_a^r \mathbf{F}(r)\cdot d\mathbf{r} + C = -\int_a^r -8F_0\left(\left(\frac{a}{r}\right)^5 - \left(\frac{a}{r}\right)^9\right)\mathbf{i}\cdot dr\mathbf{i} = F_0 a\left(1 - 2\left(\frac{a}{r}\right)^4 + \left(\frac{a}{r}\right)^8\right)$

[3 POINTS]

4. The figure shows a straight-line collision between an ion (A) and a (neutral) diatomic molecule (B-C)



- The ion and each atom are idealized as hard spheres with mass *m*.
- **The bond between atoms B and C has potential energy given in problem 3**
- For time $t<0$ atoms the bond between B and C has zero force (i.e. the distance between atoms $r=a$); and the ion A is moving to the right with speed V_0 .
- The ion A collides with atom B at time $t=0$. The collision has restitution coefficient $e=1$.

NB: the spring in the figure is used to indicate the atomic bond – it does not obey the usual force-v-length relation for a linear spring. Use the potential energy relation from problem 3,not the potential energy for a spring!)

4.1 Calculate the velocity V_1 of atom B just after the collision, in terms of V_0 (note that the bond between B and C exerts no forces during the collision)

This is a straight line perfectly elastic collision. The velocity of A is zero after impact, and the velocity of B is equal to V_0 . Note that C is stationary at this instant (no force acts on C during the impulse because there is no force in the bond).

[2 POINTS]

4.2 What is the total linear momentum of the molecule with atoms (BC) just after the impact, in terms of *m* and V_0 ?

$$\mathbf{p} = mV_0\mathbf{i}$$

[1 POINT]

4.3 What is the total kinetic energy of the molecule with atoms (BC) just after the impact, in terms of *m* and V_0 ?

$$T = \frac{1}{2}mV_0^2$$

[1 POINT]

4.4 Using energy and momentum conservation, show that the maximum separation between atoms in molecule BC following the collision is (Hint: at the instant of maximum separation both atoms B and C have the same velocity).

$$r = a\frac{1}{\left(1-\frac{1}{2}\sqrt{\frac{mV_0^2}{F_0 a}}\right)^{1/4}}$$

Note that (1) momentum is conserved; and (3) energy is conserved for the molecule with atoms BC

Let V_2 denote the velocity of C and D at the instant of maximum separation. Momentum and energy conservation give

$$mV_0 = 2mV_2$$

$$\frac{1}{2}mV_0^2 = F_0 a\left(1-2\left(\frac{a}{r}\right)^4+\left(\frac{a}{r}\right)^8\right)+mV_2^2$$

Eliminating V_2

$$0=\left(1-\frac{1}{4}\frac{mV_0^2}{F_0 a}-2\left(\frac{a}{r}\right)^4+\left(\frac{a}{r}\right)^8\right)$$

This is a quadratic equation for $(a/r)^4$ with solution

$$\left(\frac{a}{r}\right)^4 = 1\pm\frac{1}{2}\sqrt{4-4\left(1-\frac{1}{4}\frac{mV_0^2}{F_0 a}\right)} = 1-\frac{1}{2}\sqrt{\frac{mV_0^2}{F_0 a}}$$

$$\Rightarrow r = a\frac{1}{\left(1-\frac{1}{2}\sqrt{\frac{mV_0^2}{F_0 a}}\right)^{1/4}}$$

(we took the negative square root in the quadratic because we are looking for solutions with $r>a$)

[4 POINTS]

4.5 Determine the critical value of V_0 that will just break the bond between the atoms (assume that the bond breaks if $r \to \infty$ after the collision).

The bond breaks when $r \to \infty$ (so there is no more attractive force between the atoms in the molecule) which gives

$$1-\frac{1}{2}\sqrt{\frac{mV_0^2}{F_0 a}} = 0 \Rightarrow \frac{mV_0^2}{F_0 a} = 4 \Rightarrow V_0 = 2\sqrt{\frac{F_0 a}{m}}$$

[2 POINTS]