School of Engineering
Brown University

EN40: Dynamics and Vibrations

Midterm Examination
Thursday March 8 2018

NAME: ____

General Instructions

- No collaboration of any kind is permitted on this examination.
- You may bring 2 double sided pages of reference notes. No other material may be consulted
- Write all your solutions in the space provided. No sheets should be added to the exam.
- Make diagrams and sketches as clear as possible, and show all your derivations clearly. Incomplete solutions will receive only partial credit, even if the answer is correct.
- If you find you are unable to complete part of a question, proceed to the next part.

Please initial the statement below to show that you have read it

`By affixing my name to this paper, I affirm that I have executed the examination in accordance with the Academic Honor Code of Brown University. **PLEASE WRITE YOUR NAME ABOVE ALSO!**

____

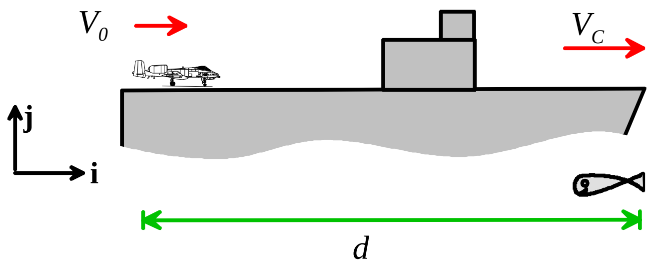
1 (10 points) _____________

2. (11 points) _____________

3. (9 points) _____________

TOTAL (30 points) ___________

1. An aircraft carrier travels with constant speed V_c. At time *t=0* an aircraft with mass *m* traveling with initial speed V_0 (relative to a stationary origin) touches down on its deck. After touchdown, the aircraft is slowed by a drag force $\mathbf{F}_D = -cv^2\mathbf{i}$, where v is the aircraft's speed (relative to a stationary origin).



1.1 Write down a formula for the distance x_c traveled by the aircraft carrier for time *t>0*, in terms of V_c

We can use the constant acceleration formula $x_c = V_c t$

[1 POINT]

1.2 Find a formula for the speed of the aircraft (relative to a stationary origin) as a function of time (for *t>0*), in terms of c, m, V_0 .

Newton's law gives $\frac{dv}{dt} = -\frac{c}{m}v^2$

Separate variables and integrate

$$\int_{V_0}^{v} \frac{dv}{v^2} = \int_0^t -\frac{c}{m}dt \Rightarrow -\frac{1}{v} + \frac{1}{V_0} = -\frac{c}{m}t \Rightarrow \frac{1}{v} = \frac{1}{V_0} + \frac{c}{m}t$$

$$\Rightarrow v = \frac{V_0 m}{m + cV_0 t}$$

[3 POINTS]

1.3 Find a formula for the distance traveled by the aircraft (relative to a stationary point) as a function of time. The following integral may be helpful

$$\int_0^t \frac{dt}{a+bt} = \frac{1}{b}\log\left(1 + \frac{b}{a}t\right)$$

From the preceding problem $\frac{dx}{dt} = \frac{V_0 m}{m + cV_0 t}$

Separate variables and integrate

$$\int_0^x dx = \int_0^t \frac{V_0 m}{m + cV_0 t}dt \Rightarrow x = \left[\frac{V_0 m}{cV_0}\log\left(m + cV_0 t\right)\right]_0^t = \frac{m}{c}\left(\log\left(m + cV_0 t\right) - \log(m)\right)$$

$$\Rightarrow x = \frac{m}{c}\log\left(1 + \frac{cV_0}{m}t\right)$$

[2 POINTS]

1.3 Hence, or otherwise, find formula for the minimum length d required for the aircraft to land without falling off the end of the deck, in terms of m, c, V_0, V_c

The aircraft comes to rest on the deck when it has the same speed as the aircraft carrier. The time taken follows from 1.2 as

$$V_c = \frac{V_0 m}{m + cV_0 t} \Rightarrow m + cV_0 t = \frac{V_0 m}{V_c} \Rightarrow t = \frac{m}{cV_0}\left(\frac{V_0}{V_c} - 1\right)$$

During this time the aircraft travels a distance

$$x_a = \frac{m}{c}\log\frac{V_0}{V_c}$$

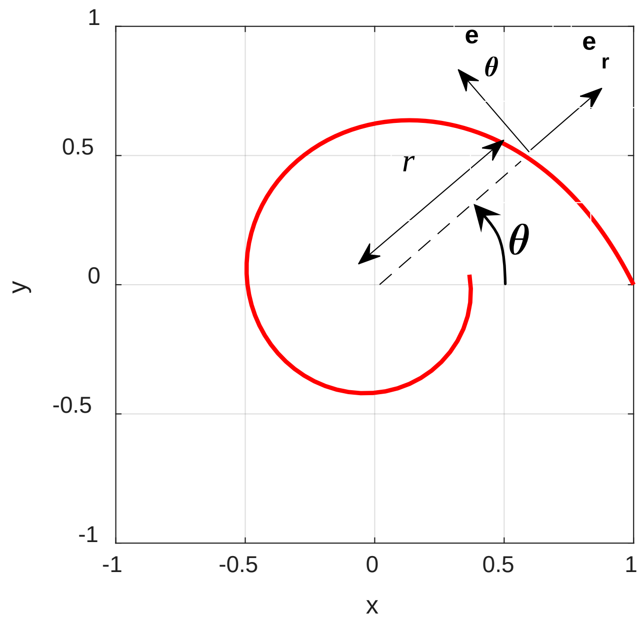
The aircraft carrier travels a distance

$$x_c = V_c t = \frac{mV_c}{cV_0}\left(\frac{V_0}{V_c} - 1\right)$$

When it reaches the end of the runway, the aircraft has traveled a distance d further than the aircraft carrier. Therefore

$$x_a = x_c + d \Rightarrow d \geq x_a - x_c = \frac{m}{c}\left(\log\frac{V_0}{V_c} + \frac{V_c}{V_0} - 1\right)$$

[4 POINTS]



2. The figure shows the trajectory of a particle with mass m in an electric field. The polar coordinates of the particle vary with time t according to the formulas

$$r = e^{-t} \qquad \theta = e^{2t} - 1$$

2.1 Find a formula for the velocity vector of the particle as a function of time, as components in the $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ basis

$$\mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta = -e^{-t}\mathbf{e}_r + 2e^{t}\mathbf{e}_\theta$$

[1 POINT]

2.2 Find a formula for the acceleration vector of the particle as a function of time, as components in the $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ basis

$$\begin{aligned}\mathbf{a} &= \left(\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta \\ &= \left(e^{-t} - e^{-t}\left(2e^{2t}\right)^2\right)\mathbf{e}_r + \left(e^{-t}4e^{2t} - 2e^{-t}2e^{2t}\right)\mathbf{e}_\theta \\ &= \left(e^{-t} - 4e^{3t}\right)\mathbf{e}_r\end{aligned}$$

[2 POINTS]

2.3 Show that the force on the particle acts in the $\mathbf{e}_r$ direction, and show that the magnitude of the force as a function of r and m is

$$|\mathbf{F}| = m\left(r - \frac{4}{r^3} \right)$$

$\mathbf{F} = m\mathbf{a}$ shows

$$\mathbf{F}(t) = m\left(e^{-t} - 4e^{3t}\right)\mathbf{e}_r = m\left(r - \frac{4}{r^3} \right)\mathbf{e}_r$$

[2 POINTS]

2.4 Find a formula for the potential energy of the force acting on the particle (your answer may include an arbitrary constant if you wish)

$$V = -\int_{r_0}^{r} m\left(r - \frac{4}{r^3} \right)\mathbf{e}_r \cdot dr\mathbf{e}_r + C = -\frac{1}{2}m\left(r^2 - r_0^2\right) - 2m\left(\frac{1}{r^2} - \frac{1}{r_0^2} \right) + C$$
$$= -m\frac{r^2}{2} - m\frac{2}{r^2}$$

[2 POINTS]

2.5 Find unit vectors normal and tangent to the path **at time t=0**.

The tangent vector is parallel to **v** so

$$\mathbf{t} = \frac{-\mathbf{e}_r + 2\mathbf{e}_\theta}{\sqrt{5}}$$

The normal vector is perpendicular to **v,** which we can find with a cross product with **k**

$$\mathbf{n} = \mathbf{k} \times \left(\frac{-\mathbf{e}_r + 2\mathbf{e}_\theta}{\sqrt{5}} \right) = \frac{1}{\sqrt{5}}(-2\mathbf{e}_r - \mathbf{e}_\theta)$$

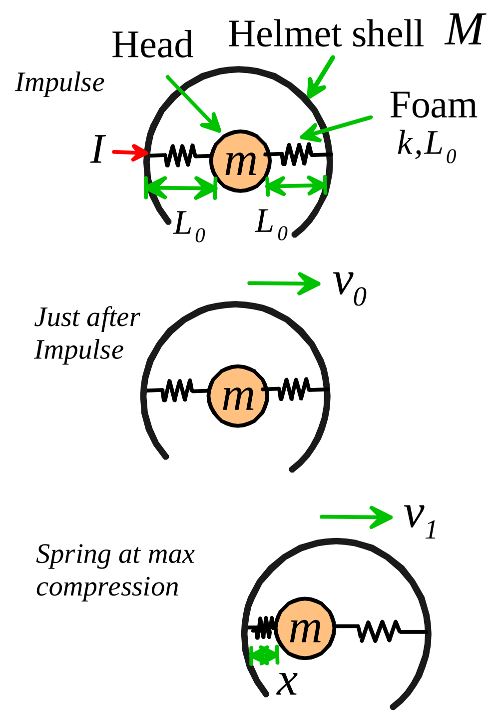
[2 POINTS]

2.6 Find the normal and tangent components of acceleration **at time t=0**.

$$a_n = \mathbf{a} \cdot \mathbf{n} = (-3)\mathbf{e}_r \cdot \frac{1}{\sqrt{5}}(-2\mathbf{e}_r - \mathbf{e}_\theta) = \frac{6}{\sqrt{5}}$$
$$a_t = \mathbf{a} \cdot \mathbf{t} = (-3)\mathbf{e}_r \cdot \left(\frac{-\mathbf{e}_r + 2\mathbf{e}_\theta}{\sqrt{5}} \right) = \frac{3}{\sqrt{5}}$$

[2 POINTS]

3. The figure shows an idealization of a head inside a protective helmet. The helmet shell has mass M, and the protective foam inside the shell is idealized as springs with stiffness k and unstretched length L_0



At time $t=0$ the helmet and head are both at rest, and the length of the springs is equal to their unstretched length.

The helmet shell is then subjected to a horizontal impulse I, which causes it to move at speed v_0 just after the impulse.

The springs exert no force on either the head or the helmet casing during the impulse. Gravity and vertical motion of the head or helmet may be neglected.

3.1. Find a formula for v_0 in terms of I and M.

$$v_0 = I / M$$

[1 POINT]

3.2. Find expressions for the total linear momentum and total kinetic energy of the system just after the impulse, in terms of I and the mass M of the shell

$$\mathbf{p}_0 = I$$

$$T_0 = \frac{1}{2} M v_0^2 = \frac{I^2}{2M}$$

[2 POINTS]

3.3. Consider the system at the instant when the foam on the left of the head is compressed to its smallest thickness (i.e. x is a minimum). Using energy and/or momentum conservation, find a formula for x at this instant. Assume that the foam remains in contact with the head on both sides (so one spring is stretched), as indicated in the figure.

Energy and momentum are conserved, and the casing and head have the same speed at the instant of max compression, so

$$\frac{1}{2}(M+m)v_1^2 + k(L_0 - x)^2 = \frac{I^2}{2M}$$

$$(M+m)v_1 = I$$

$$\Rightarrow k(L_0 - x)^2 = \frac{I^2}{2M} - \frac{I^2}{2(M+m)} = \frac{mI^2}{2M(M+m)}$$

$$\Rightarrow x = L_0 - I\sqrt{\frac{m}{2kM(M+m)}}$$

[3 POINTS]

3.4. Hence, find a formula for the minimum foam thickness L_0 necessary to prevent the head from striking the helmet shell, in terms of m, M, I,k

$$L_0 = I\sqrt{\frac{m}{2kM(M+m)}}$$

[1 POINT]

3.5. Find a formula for the total impulse exerted on the head between $t=0$ and the instant of maximum foam compression, in terms of I,M,m.

The impulse on the head is equal to its change in momentum. Thus

$$\mathbf{p} = mv_1\mathbf{i} = \frac{m}{M+m}I$$

[2 POINTS]