1.

Solutions HW 7
Rigid Body Kin.

$$\underline{V}_B = \underline{V}_A + (\underline{\omega}_{AB} \times \underline{r}_{B/A})$$

$$= 0 + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 4 \\ 2 & 12 & 0 \end{vmatrix} = (-48\hat{i} + 8\hat{j}) \text{ cm/s}$$

$$\underline{V}_C = \underline{V}_B + (\underline{\omega}_{BC} \times \underline{r}_{C/B})$$

$$\underline{V}_C = (-48\hat{i} + 8\hat{j}) + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega_{BC} \\ 16 & -2 & 0 \end{vmatrix}$$

$$* \; \underline{V}_C = (-48 + 2\omega_{BC})\hat{i} + (8 + 16\omega_{BC})\hat{j}$$

$$\underline{V}_D = \underline{V}_C + (\underline{\omega}_{CD} \times \underline{r}_{D/C})$$

$$0 = \underline{V}_C + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega_{CD} \\ 4 & 3 & 0 \end{vmatrix} = \underline{V}_C + (-3\omega_{CD})\hat{i} + 4\omega_{CD}\hat{j}$$

$$* \; \underline{V}_C = 3\omega_{CD}\hat{i} - 4\omega_{CD}\hat{j} \implies \begin{matrix} 2 \text{ eqns } (*) \\ 2 \text{ unknowns} \end{matrix}$$

$$(3\omega_{CD} = -48 + 2\omega_{BC})\hat{i}$$

$$(-4\omega_{CD} = 8 + 16\omega_{BC})\hat{j}$$

$$\boxed{\begin{aligned} \omega_{BC} &= 3 \text{ rad/s} \\ \omega_{CD} &= -14 \text{ rad/s} \end{aligned}}$$

2.1. Front gear is rigidly attached to petals

$\alpha_p = \alpha_1$

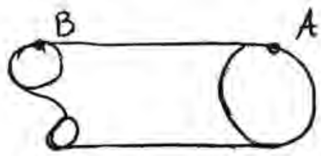
$\omega_p = \omega_1$

Integrating $\alpha_p = \begin{cases} t/2 & 0 \leq t < 2 \\ 1 & 2 \leq t < 12 \\ -t/2 + 7 & 12 \leq t \leq 14 \end{cases}$

We obtain $\omega_1 = \begin{cases} t^2/4 & 0 \leq t < 2 \\ t - 1 & 2 \leq t < 12 \\ -t^2/4 + 7t - 37 & 12 \leq t \leq 14 \end{cases}$

see attached for plots

2.2.



At A & B, velocities must be equal: $V_A = V_B$

$r_2 \omega_2 = r_1 \omega_1$

$\omega_2 = \left(\frac{r_1}{r_2}\right) \omega_1$

$\omega_2 = 8/3\ \omega_1$ see attached plot

2.3. Number of Revolutions is found by integrating $\omega_p(t)$

$$\theta_p(t) = \begin{cases} t^3/12 & 0 \leq t < 2 \\ t^2/2 - t + 2/3 & 2 \leq t < 12 \\ -t^3/12 + 7/2\, t^2 - 37t + \frac{434}{3} & 12 \leq t \leq 14 \end{cases}$$

$\theta_p(t=14\,sec) = 84$ radians

$84/2\pi = 13.4$ Revolutions

2.4. Distance travelled is revolutions of rear wheel $\times$ $2\pi r_w$ (circumference).

$\theta_w = \theta_2 = \int \omega_2\,dt$ and $\omega_2 = 8/3\,\omega_1$ so

$$\theta_w(t) = 8/3 \int \omega_1\,dt = 8/3 \cdot \theta_1(t)$$

$$\theta_w(t=14\,sec.) = 224 \text{ radians}$$

$$\frac{224}{2\pi} = 35.7 \text{ revolutions}$$

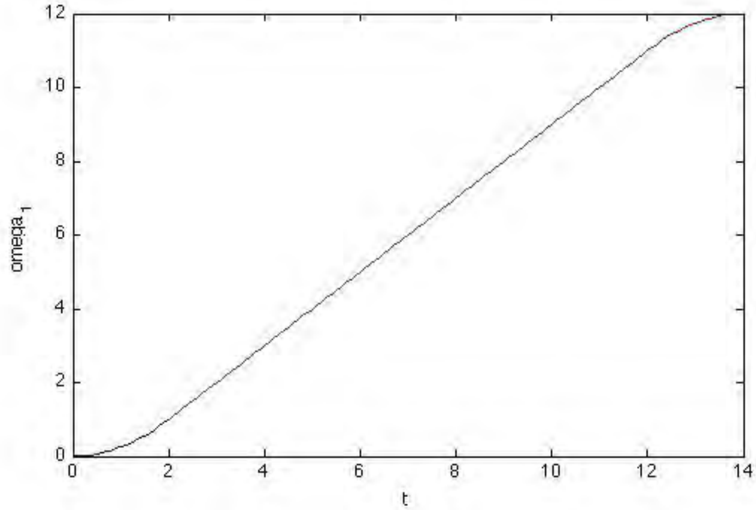
$$35.7\,(2\pi r_w) = 74 \text{ meters}$$

2.5. The forward velocity is $v_o = \omega_w r_w$

$$v_o = r_w \cdot \omega_2$$

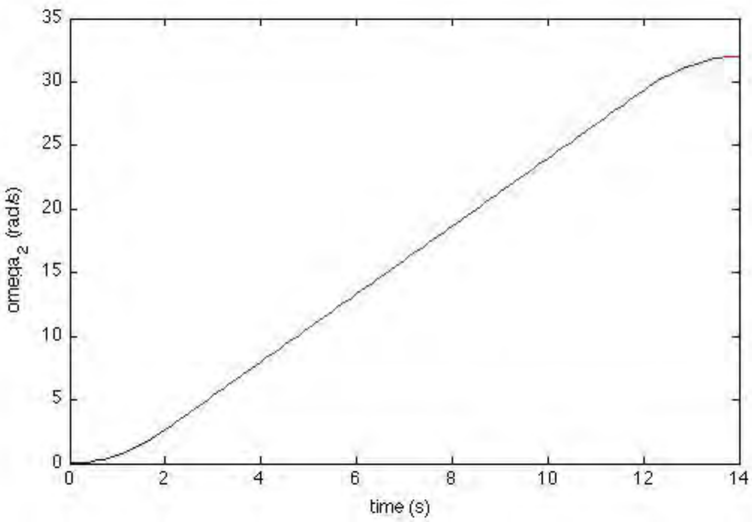
see attached plot

2.1

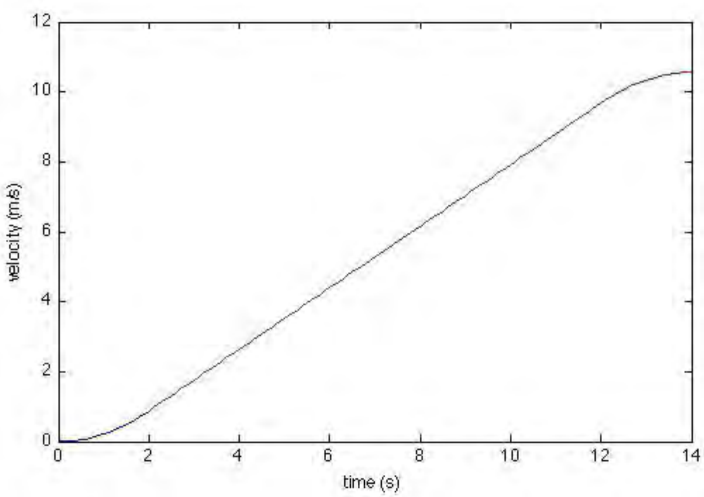


12
10
8
6
4
2
0
omega$_1$
0
2
4
6
8
10
12
14
t

2.2



35
30
25
20
15
10
5
0
omega$_2$ (rad/s)
0
2
4
6
8
10
12
14
time (s)



12
10
8
6
4
2
0
velocity (m/s)
0
2
4
6
8
10
12
14
time (s)

2.5

Helmet Impact

3.1
$$\underline{a}_1 = \underline{a}_{CM} + (\underline{\alpha} \times \underline{r}_1) + (\underline{\omega} \times (\underline{\omega} \times \underline{r}_1))$$

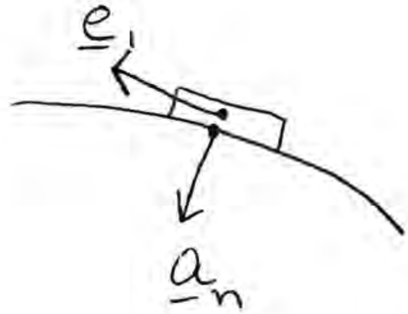
3.2.
$$a_1 = \left[\underline{a}_{CM} + (\underline{\alpha} \times \underline{r}_1) + (\underline{\omega} \times (\underline{\omega} \times r_1))\right] \cdot \underline{e}_1$$

.3. unknowns: 3 components of $\underline{a}_{CM}$ +
3 components of $\underline{\alpha}$ = 6!

We need 6 equations and 6 accelerometers!

$$a_1 = \dots$$
$$a_2 = \dots$$
$$\vdots$$
$$a_6 = \dots$$

3.4.



$$\underline{a}_t = \underline{\alpha} \times \underline{r}_1$$
$$\underline{a}_n = \underline{\omega} \times (\underline{\omega} \times \underline{r}_1)$$

If $\underline{e}_1$ is perpendicular to $\underline{a}_n$ Then $\underline{e}_1 \cdot \underline{a}_n = 0$. Thus we can neglect $\underline{\omega} \times (\underline{\omega} \times \underline{r}_1)$

$$\boxed{a_1 = \left[\underline{a}_{CM} + (\underline{\alpha} \times \underline{r}_1)\right] \cdot \underline{e}_1}$$

3.5.
$$a_1 = (a_x + \alpha_y r_z - \alpha_z r_y) e_x + (a_y - \alpha_x r_z + \alpha_z r_x) e_y + (a_z + \alpha_x r_y - \alpha_y r_x) e_z$$

3.6 $\underline{A}\,\underline{X} = \underline{B}$ for Matlab

$$\underline{X} = \begin{bmatrix} a_x \\ a_y \\ a_z \\ \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} \quad \underline{B} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}$$

$\underline{A}$ = 6×6 matrix of coefficients r_n & $\underline{e}_n$ for $n = 1 \to 6$

see attached code

$|\underline{a}_{cm}| = 404\ m/s^2$ OR 41 g's

$a_x = 400\ m/s^2$
$a_y = -30\ m/s^2$
$a_z = 50\ m/s^2$

$\alpha = (698\,\hat{i} + 5000\,\hat{j} - 12000\,\hat{k})\ rad/s^2$

HW #7 Problem #3

(a) $$\underline{r}_D = R\,\underline{i} + L\,\underline{k}$$

$$\underline{r}_E = R\cos\left(\frac{2\pi}{3}\right)\underline{i} + R\sin\left(\frac{2\pi}{3}\right)\underline{j} + L\,\underline{k}$$

$$\underline{r}_F = R\cos\left(\frac{4\pi}{3}\right)\underline{i} + R\sin\left(\frac{4\pi}{3}\right)\underline{j} + L\,\underline{k}$$

(b) $$\underline{r}_a = R\cos\theta\,\underline{i} + R\sin\theta\,\underline{j} + z\,\underline{k}$$

$$\underline{r}_b = R\cos\left(\frac{2\pi}{3}+\theta\right)\underline{i} + R\sin\left(\frac{2\pi}{3}+\theta\right)\underline{j} + z\,\underline{k}$$

$$\underline{r}_c = R\cos\left(\frac{4\pi}{3}+\theta\right)\underline{i} + R\sin\left(\frac{4\pi}{3}+\theta\right)\underline{j} + z\,\underline{k}$$

(c) $$\underline{r}_{aD} = \underline{r}_D - \underline{r}_a = R(1-\cos\theta)\,\underline{i} - R\sin\theta\,\underline{j} + (L-z)\underline{k}$$

$$\left|\underline{r}_{aD}\right|^2 = L^2$$

$$\Rightarrow \quad R^2(1-\cos\theta)^2 + R^2\sin^2\theta + (L-z)^2 = L^2$$

$$R^2 + R^2\cos^2\theta - 2R^2\cos\theta + R^2\sin^2\theta + \cancel{L^2} + z^2 - 2Lz = \cancel{L^2}$$

$$\Rightarrow \quad 2R^2(1-\cos\theta) + z(z-2L) = 0$$

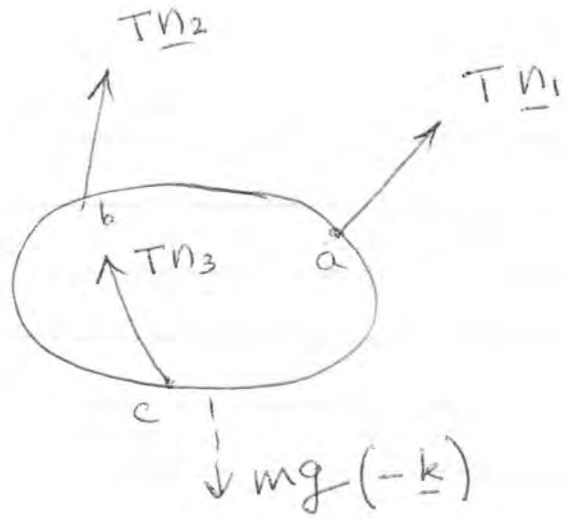
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \qquad \text{For small } \theta,\ \cos\theta = 1 - \frac{\theta^2}{2}$$

Also, for small values of θ, $z \ll 2L \Rightarrow z - 2L \simeq -2L$

$$2R^2\frac{\theta^2}{2} - 2Lz = 0 \Rightarrow z \approx \frac{R^2\theta^2}{2L}$$

(d) $$\underline{n}_1 = \frac{\underline{r}_D - \underline{r}_a}{L}, \quad \underline{n}_2 = \frac{\underline{r}_E - \underline{r}_b}{L}, \quad \underline{n}_3 = \frac{\underline{r}_F - \underline{r}_c}{L}$$

(e)



(f) Net Force = (acceleration of the CoM) * m

$$T\underline{n}_1 + T\underline{n}_2 + T\underline{n}_3 - mg\underline{k} = m\left[a_x \underline{i} + a_y \underline{j} + \frac{d^2 z}{dt^2}\underline{k}\right]$$

$$\Rightarrow \frac{T}{L}\left[(\underline{r}_D + \underline{r}_E + \underline{r}_F) - (\underline{r}_a + \underline{r}_b + \underline{r}_c)\right] - mg\,\underline{k} = m\left[a_x\underline{i} + a_y\underline{j} + \frac{d^2 z}{dt^2}\underline{k}\right]$$

(g) $$\frac{\underline{r}_a + \underline{r}_b + \underline{r}_c}{3} = z\,\underline{k}$$

$$\frac{\underline{r}_D + \underline{r}_E + \underline{r}_F}{3} = L\,\underline{k}$$

$$\Rightarrow m\left[a_x\,\underline{i} + a_y\,\underline{j} + \frac{d^2 z}{dt^2}\underline{k}\right] = \frac{T}{L}\left[3L\,\underline{k} - 3z\,\underline{k}\right] - mg\,\underline{k}$$

$$= \left[3T\left(1 - \frac{z}{L}\right) - mg\right]\underline{k}$$

(h) For small values of z, $z \approx 0$, $\frac{d^2 z}{dt^2} \approx 0$

$$\Rightarrow 3T - mg = 0 \Rightarrow T = \frac{mg}{3}$$

(i) $\underbrace{\sum \underline{M}_{com}}_{\uparrow \text{Net moment about the center of mass.}} = I\underline{\alpha}$

$$\Rightarrow I\frac{d^2\theta}{dt^2}\underline{k} = (\underline{r}_a - z\underline{k}) \times \frac{T}{L}(\underline{r}_D - \underline{r}_a) + (\underline{r}_b - z\underline{k}) \times \frac{T}{L}(\underline{r}_E - \underline{r}_b) + (\underline{r}_c - z\underline{k}) \times \frac{T}{L}(\underline{r}_F - \underline{r}_c)$$

$$= \frac{T}{L}\Big[\underline{r}_a \times (\underline{r}_D - \underline{r}_a) - z\underline{k} \times (\underline{r}_D - \underline{r}_a) + \underline{r}_b \times (\underline{r}_E - \underline{r}_b) - z\underline{k} \times (\underline{r}_E - \underline{r}_b) + \underline{r}_c \times (\underline{r}_F - \underline{r}_c) - z\underline{k} \times (\underline{r}_F - \underline{r}_c)\Big]$$

$$= \frac{T}{L}\Big[\underline{r}_a \times \underline{r}_D - \cancelto{0}{\underline{r}_a \times \underline{r}_a} - z\underline{k} \times \underline{r}_D + z\underline{k} \times \underline{r}_a + \underline{r}_b \times \underline{r}_E - \cancelto{0}{\underline{r}_b \times \underline{r}_b} - z\underline{k} \times \underline{r}_E + z\underline{k} \times \underline{r}_b + \underline{r}_c \times \underline{r}_F - \cancelto{0}{\underline{r}_c \times \underline{r}_c} - z\underline{k} \times \underline{r}_F + z\underline{k} \times \underline{r}_c\Big]$$

$$= \frac{T}{L}\Big[(\underline{r}_a \times \underline{r}_D) + (\underline{r}_b \times \underline{r}_E) + (\underline{r}_c \times \underline{r}_F) + z\underline{k} \times (\underline{r}_a + \underline{r}_b + \underline{r}_c) - z\underline{k} \times (\underline{r}_D + \underline{r}_E + \underline{r}_F)\Big]$$

$$= \frac{T}{L}\Big[(\underline{r}_a \times \underline{r}_D) + (\underline{r}_b \times \underline{r}_E) + (\underline{r}_c \times \underline{r}_F) + \cancelto{0}{z\underline{k} \times 3z\underline{k}} - \cancelto{0}{z\underline{k} \times 3L\underline{k}}\Big] \quad \text{——} (*)$$

$$\underline{r}_a \times \underline{r}_D = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ R\cos\theta & R\sin\theta & z \\ R & 0 & L \end{vmatrix} = LR\sin\theta\,\underline{i} - R(L\cos\theta - z)\,\underline{j} + R^2\sin\theta\,\underline{k}$$

Since the LHS of the above equation $(*)$ has only $\underline{k}$, we need to consider only the $\underline{k}$ component on RHS.

$$\underline{r}_b \times \underline{r}_E = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ R\cos\left(\frac{2\pi}{3}+\theta\right) & R\sin\left(\frac{2\pi}{3}+\theta\right) & z \\ R\cos\left(\frac{2\pi}{3}\right) & R\sin\left(\frac{2\pi}{3}\right) & L \end{vmatrix}$$

$$= \underline{i}\left(R^2\cos\left(\frac{2\pi}{3}+\theta\right)\sin\frac{2\pi}{3} - R^2\sin\left(\frac{2\pi}{3}+\theta\right)\cos\frac{2\pi}{3}\right)$$

$$- \underline{j}\left(LR\cos\left(\frac{2\pi}{3}+\theta\right) - Rz\cos\frac{2\pi}{3}\right) + \underbrace{\underline{k}\left(R^2\cos\left(\frac{2\pi}{3}+\theta\right)\sin\frac{2\pi}{3} - R^2\sin\left(\frac{2\pi}{3}+\theta\right)\cos\frac{2\pi}{3}\right)}_{-\underline{k}\,R^2\sin\theta}$$

(using the identity: $\sin(x+y) = \sin x\cos y + \cos x \sin y$)

Similarly

$$\underline{r}_c \times \underline{r}_F = \underline{i}\left[R^2\cos\left(\frac{4\pi}{3}+\theta\right)\sin\frac{4\pi}{3} - R^2\sin\left(\frac{4\pi}{3}+\theta\right)\cos\frac{4\pi}{3}\right]$$

$$- \underline{j}\left[LR\cos\left(\frac{4\pi}{3}+\theta\right) - Rz\cos\frac{4\pi}{3}\right] + \underbrace{\underline{k}\left[R^2\cos\left(\frac{4\pi}{3}+\theta\right)\sin\frac{4\pi}{3} - R^2\sin\left(\frac{4\pi}{3}+\theta\right)\cos\frac{4\pi}{3}\right]}_{-\underline{k}\,R^2\sin\theta}$$

You can verify that the sum of the $\underline{i}$ components and $\underline{j}$ components add upto zero.

$$\Rightarrow I\frac{d^2\theta}{dt^2} = \left(-3R^2\sin\theta\right)\frac{T}{L} \Rightarrow \boxed{I\frac{d^2\theta}{dt^2} + \frac{3R^2T}{L}\sin\theta = 0}$$

(j) For small values of θ, $\sin\theta \sim \theta$

$$\Rightarrow \omega_n = \sqrt{\frac{3R^2T}{IL}} \quad \text{———} \; *$$

(k) Let's call the time period from (j) above, i.e.

the approximate time period $T_{app} = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{3R^2T}{IL}}}$ — (2)

Now, we can calculate the actual time period (T_{real}) for a given value of θ_0 by solving the eom (1) with MATLAB. To write MATLAB code, we need numerical values for R, T, I, L. We can proceed in one of 2 ways.

(i) Assume arbitrary values for R, T, I, L, ..

Get T_{app} from eq (2) above

Solve EOM with MATLAB

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{3R^2T}{IL}\sin\theta \end{bmatrix}$$, Initial conditions $[\theta_0, 0]$

Get T_{real} as a function of θ_0

Find the range of θ_0 for which T_{app} & T_{real} are within 5% and 1% of each other.

(ii) Alternatively, we can eliminate the physical parameters by normalizing the time as

$\tau = \dfrac{t}{T_{app}}$. The the EOM becomes

$$\frac{1}{T_{app}^2}\frac{d^2\theta}{d\tau^2} + \frac{4\pi^2}{T_{app}^2}\sin\theta = 0 \Rightarrow \boxed{\frac{d^2\theta}{d\tau^2} + 4\pi^2\sin\theta = 0}$$

This equation is now in terms of non-dimensional quantities only. Linearize it : $\dfrac{d^2\theta}{d\tau^2} + 4\pi^2\theta = 0$

$\Rightarrow$ Approximate time period $\tau_{app} = 1$

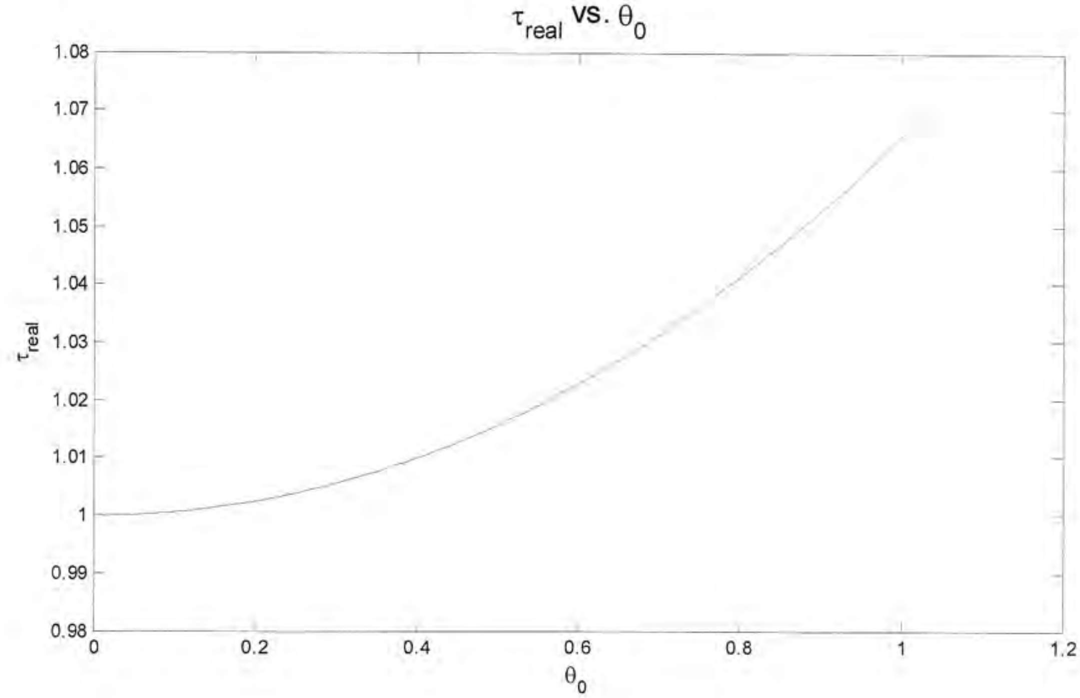
Use MATLAB to find τ_{real} as a function of θ_0.

Find the range of θ_0 for which τ_{app} & τ_{real} are within 5% & 1% of each other.

For 5% error, $\theta_0 = 0.8692$ radians

For 1% error, $\theta_0 = 0.3979$ radians.

```
function hw7problem3
    close all;

    % write a for loop to solve the equation of motion (eom) for
different
    % values of theta_0 between 0 and pi/3.
    for i=1:100
        theta0(i)=pi/3*i/100;
        ic=[theta0(i),0];
        options=odeset('Event',@detect_zero_crossing);
        [t,w,tevent,wevent,index]=ode45(@eom,[0,5],ic,options);
        % time difference between two successive events is t_real
        t_real(i)=tevent(3)-tevent(2);
    end

    plot(theta0,t_real)

    % following is the code to detect the range of theta_0 for which
    % t_real is within 5% and 1% of t_app

    for i=1:100
        if (t_real(i)<1.05)
            theta0_5=theta0(i);
        end
        if (t_real(i)<1.01)
            theta0_1=theta0(i);
        end
    end

    % write the values of theta_0 for 5% error adn 1% error to the
screen.
    theta0_5
    theta0_1

    % function of the eom in terms of the normalized time "tau"
    function dwdt=eom(t,w)
        dwdt=[w(2);-4*pi^2*sin(w(1))];
    end

    % event function to detect when theta crosses zero with positive
slope
    function [value,stop,dir]=detect_zero_crossing(t,w)
        value=w(1);
        stop=0;
        dir=1;
    end

end
```


τ_{real} vs. θ_0
1.08
1.07
1.06
1.05
1.04
1.03
1.02
1.01
1
0.99
0.98
τ_{real}
0
0.2
0.4
0.6
0.8
1
1.2
θ_0