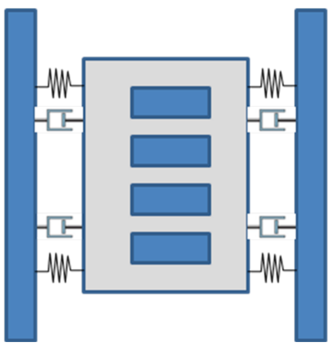
School of Engineering
Brown University

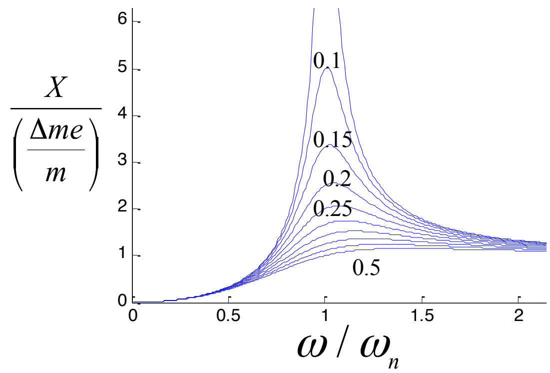
EN40: Dynamics and Vibrations

Homework 7: Rigid Body Kinematics

1. FORCED VIBRATION OF ENGINE IDLING

The Subaru Legacy has an unusual horizontally-aligned four-cylinder engine. The engine is connected to lateral motor mounts by 4 springs and 4 dashpots, as shown in the figure. When idling, slightly differences in the firing of the individual cylinders lead to an effective *rotor forcing* of the engine, with an effective mass imbalance of $e\Delta m = 0.4kgm$ at a frequency corresponding to $600/\pi$ RPM. The total mass of the engine is M=200kg. Brand new, each spring has stiffness k=12800N/m and each damper has damping coefficient λ=400 N-s/m.





1.1 What are the natural frequency and damping coefficient ς for the engine?

From formulas $\omega_n = \sqrt{k/m} = \sqrt{4 \times 12800/200} = 16 rad/s$.
The damping coefficient is $\zeta = c/2\sqrt{km} = 4 \times 400 / 2\sqrt{4 \times 12800 \times 200} = 1/4$

(2 POINTS)

1.2 What is the typical steady-state amplitude of the lateral vibrations of the engine?

The forcing frequency is $\omega = 2\pi \times 600/\pi/60 = 20 rad/s \Rightarrow \omega/\omega_n = 20/16$. **Doing the calculation with the formula,**

$X/(\Delta me/m) = (\omega/\omega_n)^2 / \sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + 4\zeta^2\omega^2/\omega_n^2} = 1.86 \Rightarrow X = 1.86\Delta me/m = 0.0037m$. **Or more quickly reading off the graph** $X/(\Delta me/m)$ **is about 2…**

(3 POINTS)

1.3 The main problem is not the engine vibration, but the forces caused on the attachment points to the body of the car. These attachment fixtures fatigue under load. What is the approximate amplitude of the force exerted on one damper attachment point for the new car?

The force is $c dx/dt = cX\omega\cos(\omega t + \phi) \approx 29.6\cos(\omega t + \phi)N$

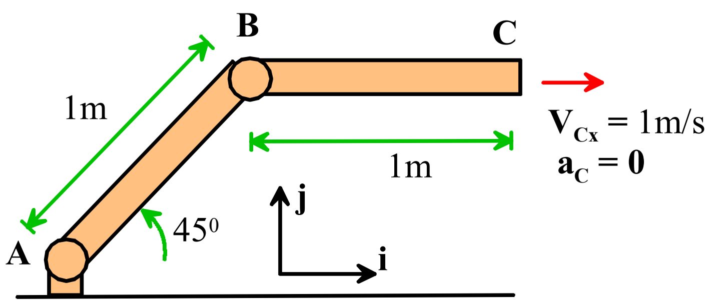
(3 POINTS)

1.4 As the car ages, the spring stiffness gets smaller. How does this change the vibration amplitude? (increase, decrease, the same?)

This decreases ω_n so the operating point shifts to the right on the curve, and the amplitude decreases.

(2 POINTS)

2. ROBOT ARM KINEMATICS



The figure shows a robot arm. Point C on the arm is required to move horizontally with constant speed 1m/s. This is accomplished by rotating links AB and BC with appropriate angular speeds ω_{AB}, ω_{BC} and angular accelerations α_{AB}, α_{BC}. The goal of this problem is to calculate values for ω_{AB}, ω_{BC}, α_{AB}, α_{BC} at the instant shown.

2.1 Determine formulas for the velocity vectors $\mathbf{v}_B, \mathbf{v}_C$ of points B and C, in terms of ω_{AB}, ω_{BC}.

Applying the rigid body kinematics formula gives

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A} = \omega_{AB}\mathbf{k} \times (\mathbf{i}+\mathbf{j})/\sqrt{2} = \omega_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2}$$

$$\mathbf{v}_C = \mathbf{v}_B + \omega_{BC}\mathbf{k} \times \mathbf{i} = \omega_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2} + \omega_{BC}\mathbf{j}$$

(3 POINTS)

2.2 Determine formulas for the acceleration vectors $\mathbf{a}_B, \mathbf{a}_C$ of points B and C in terms of $\alpha_{AB}, \alpha_{BC}, \omega_{AB}, \omega_{BC}$.

$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \alpha_{AB}\mathbf{k} \times (\mathbf{i}+\mathbf{j})/\sqrt{2} - \omega_{AB}\mathbf{k} \times \omega_{AB}\mathbf{k} \times (\mathbf{i}+\mathbf{j})/\sqrt{2} \\ &= \alpha_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2} - \omega_{AB}^2(\mathbf{i}+\mathbf{j})/\sqrt{2}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_B + \alpha_{BC}\mathbf{k} \times \mathbf{i} - \omega_{BC}\mathbf{k} \times \omega_{BC}\mathbf{k} \times \mathbf{i} \\ &= \alpha_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2} - \omega_{AB}^2(\mathbf{i}+\mathbf{j})/\sqrt{2} + \alpha_{BC}\mathbf{j} - \omega_{BC}^2\mathbf{i}\end{aligned}$$

(3 POINTS)

2.3 Hence, calculate the required values of $\omega_{AB}, \omega_{BC}, \alpha_{AB}, \alpha_{BC}$

We know that $\mathbf{v}_C = \mathbf{i}$. Using the **i** and **j** components of 12.1 gives two equations for ω_{AB}, ω_{BC}

$$-\omega_{AB} = \sqrt{2} rad/s \qquad \omega_{AB}/\sqrt{2} + \omega_{BC} = 0 \Rightarrow \omega_{BC} = 1 rad/s$$

We also know that $\mathbf{a}_C = \mathbf{0}$ which gives

$$\alpha_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2}-\omega_{AB}^2(\mathbf{i}+\mathbf{j})/\sqrt{2}+\alpha_{BC}\mathbf{j}-\omega_{BC}^2\mathbf{i}=\mathbf{0}$$
$$\Rightarrow \alpha_{AB}(-\mathbf{i}+\mathbf{j})/\sqrt{2}+\alpha_{BC}\mathbf{j}-2(\mathbf{i}+\mathbf{j})/\sqrt{2}-\mathbf{i}=\mathbf{0}$$
$$\Rightarrow \alpha_{AB}=-\left(2+\sqrt{2}\right)\ rad/s^2$$
$$\alpha_{BC}=\frac{1}{4\sqrt{2}}-\frac{\alpha_{AB}}{\sqrt{2}}=\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(2+\sqrt{2}\right)=1+2\sqrt{2}\,rad/s^2$$

(4 POINTS)

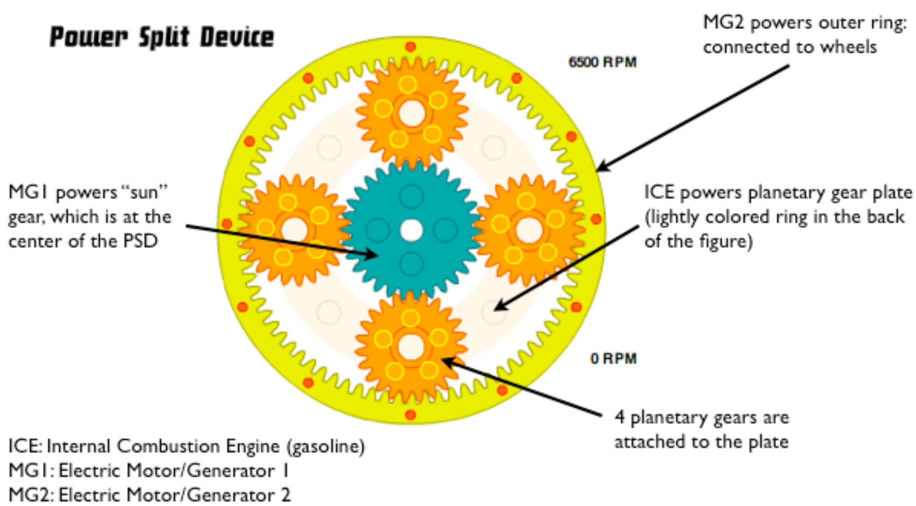
3. PRIUS POWER SPLIT DEVICE (PSD)

In class we saw a demonstration of the Prius' Planetary Gear Set (http://eahart.com/prius/psd/).

3.1 At the lowest speeds (<42 mph), the ICE does not have to provide any power. Which components of the PSD are spinning, and in which direction?

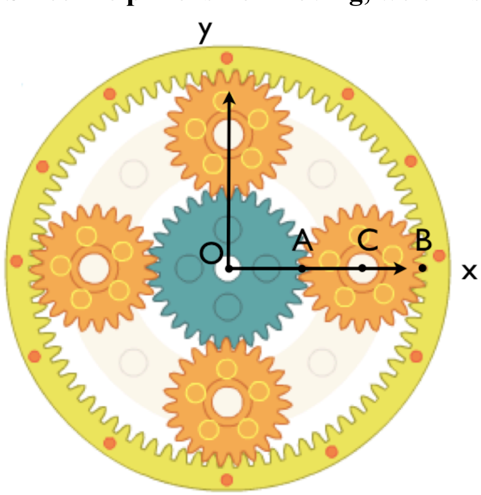
Gear plate – NOT moving.
Outer Ring CW (-)
Sun Ring CCW (+)
Planets CW (-)

(1 POINT)



3.2 In this configuration, what is the gear ratio between the sun gear (rotational speed ω_s) and the outer ring (rotational speed ω_r) in terms of the radius of the sun gear, r_s, and the radius of the planetary gear, r_p?

Since the plate is not moving, we can say:



$\mathbf{v_A = \omega_s r_s = \omega_p r_p}$
$\mathbf{v_B = \omega_r r_r = \omega_p r_p}$
Thus, $\mathbf{\omega_s r_s = \omega_r r_r}$
From geometry, $\mathbf{r_r = r_s + 2r_p}$

$\mathbf{\omega_s = [(r_s+2r_p)/r_s]\omega_r}$
Gear Ratio: $\mathbf{[(r_s+2r_p)/r_s]}$
Also accepted: $\mathbf{[r_s/(r_s+2r_p)]}$

(3 POINTS)

3.3 Now the outer ring is not rotating (the car is not moving) but the ICE engine continues to run! What is the gear ratio between the sun gear and the gear plate (rotational speed ω_{pp}) in terms of r_s and r_p?

$\mathbf{v_A = \omega_s r_s}$ **as before (because O remains fixed)**

From our kinematics equations:
$y_B = y_C + |r_{B/C}| \sin\theta$
$v_B = v_C + r_p\, d\theta/dt \cos\theta$
however let's examine the point at which θ=0:
$v_B = v_C + r_p\, d\theta/dt$
$v_B = \omega_{pp}(r_s+r_p) + r_p\, d\theta/dt$

We can also write down how B depends on A:
$y_B = y_A + |r_{B/A}| \sin\theta$
$v_B = v_A + 2r_p\, d\theta/dt \cos\theta$
however let's examine the point at which θ=0:
$v_B = v_A + 2r_p\, d\theta/dt$
$v_B = \omega_s r_s + 2r_p\, d\theta/dt$

If the outer ring remains fixed: $v_B = \omega_r r_r = 0$
$-2\omega_{pp}(r_s+r_p) = 2\, r_p\, d\theta/dt$
$\omega_s r_s = -\, 2r_p\, d\theta/dt$

eliminating dθ/dt:
$2\omega_{pp}(r_s+r_p) = \omega_s r_s$
$\omega_s = [2(r_s+r_p) / r_s]\omega_{pp}$

gear ratio: $[2(r_s+r_p)/ r_s]$ **(or inverse)**

(3 POINTS)

3.4 For the configuration where all components are rotating, derive a relationship between ω_r, ω_s, and ω_{pp} in terms of r_s and r_p.

We can use the same solution method as above except B is not stationary:
Starting with these 2 equations from 3.3:
$v_B = \omega_{pp}(r_s+r_p) + r_p\, d\theta/dt$
$v_B = \omega_s r_s + 2r_p\, d\theta/dt$

Eliminate dθ/dt and substitute $v_B = r_r\, \omega_r = (r_s+2r_p)\, \omega_r$
$(r_s+2r_p)\, \omega_r = 2(r_s+r_p)\omega_{pp} - r_s\, \omega_s$

(3 POINTS)

4. TRIFILAR PENDULUM (1 POINT EACH PART – SEE SCANNED SOLNS ATTACHED)

HW #7 Problem #3

(a) $\underline{r}_D = R\,\underline{i} + L\,\underline{k}$

$\underline{r}_E = R\cos\left(\frac{2\pi}{3}\right)\underline{i} + R\sin\left(\frac{2\pi}{3}\right)\underline{j} + L\,\underline{k}$

$\underline{r}_F = R\cos\left(\frac{4\pi}{3}\right)\underline{i} + R\sin\left(\frac{4\pi}{3}\right)\underline{j} + L\,\underline{k}$

(b) $\underline{r}_a = R\cos\theta\,\underline{i} + R\sin\theta\,\underline{j} + z\,\underline{k}$

$\underline{r}_b = R\cos\left(\frac{2\pi}{3}+\theta\right)\underline{i} + R\sin\left(\frac{2\pi}{3}+\theta\right)\underline{j} + z\,\underline{k}$

$\underline{r}_c = R\cos\left(\frac{4\pi}{3}+\theta\right)\underline{i} + R\sin\left(\frac{4\pi}{3}+\theta\right)\underline{j} + z\,\underline{k}$

(c) $\underline{r}_{aD} = \underline{r}_D - \underline{r}_a = R(1-\cos\theta)\underline{i} - R\sin\theta\,\underline{j} + (L-z)\underline{k}$

$|\underline{r}_{aD}|^2 = L^2$

$\Rightarrow\ R^2(1-\cos\theta)^2 + R^2\sin^2\theta + (L-z)^2 = L^2$

$R^2 + R^2\cos^2\theta - 2R^2\cos\theta + R^2\sin^2\theta + \cancel{L^2} + z^2 - 2Lz = \cancel{L^2}$

$\Rightarrow\ 2R^2(1-\cos\theta) + z(z-2L) = 0$

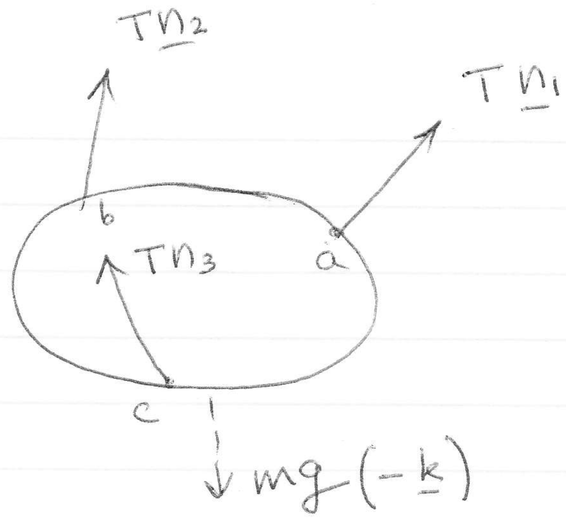
$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$ For small θ, $\cos\theta = 1 - \frac{\theta^2}{2}$

Also, for small values of θ, $z \ll 2L \Rightarrow z - 2L \simeq -2L$

$2R^2\frac{\theta^2}{2} - 2Lz = 0 \Rightarrow z \approx \frac{R^2\theta^2}{2L}$

(d) $\underline{n}_1 = \frac{\underline{r}_D - \underline{r}_a}{L}$, $\underline{n}_2 = \frac{\underline{r}_E - \underline{r}_b}{L}$, $\underline{n}_3 = \frac{\underline{r}_F - \underline{r}_c}{L}$

(e)



(f) Net Force = (acceleration of the COM) * m

$$T\underline{n}_1 + T\underline{n}_2 + T\underline{n}_3 - mg\underline{k} = m\left[a_x \underline{i} + a_y \underline{j} + \frac{d^2 z}{dt^2}\underline{k}\right]$$

$$\Rightarrow \frac{T}{L}\left[(\underline{r}_D + \underline{r}_E + \underline{r}_F) - (\underline{r}_a + \underline{r}_b + \underline{r}_c)\right] - mg\underline{k} = m\left[a_x\underline{i} + a_y\underline{j} + \frac{d^2 z}{dt^2}\underline{k}\right]$$

(g) $$\frac{\underline{r}_a + \underline{r}_b + \underline{r}_c}{3} = z\underline{k}$$

$$\frac{\underline{r}_D + \underline{r}_E + \underline{r}_F}{3} = L\underline{k}$$

$$\Rightarrow m\left[a_x\underline{i} + a_y\underline{j} + \frac{d^2 z}{dt^2}\underline{k}\right] = \frac{T}{L}\left[3L\underline{k} - 3z\underline{k}\right] - mg\underline{k}$$

$$= \left[3T\left(1 - \frac{z}{L}\right) - mg\right]\underline{k}$$

(h) For small values of z, $z \approx 0$, $\frac{d^2 z}{dt^2} \approx 0$

$$\Rightarrow 3T - mg = 0 \Rightarrow T = \frac{mg}{3}$$

(i) $\underbrace{\sum \underline{M}_{COM}}_{\uparrow} = I\underline{\alpha}$

Net moment about the center of mass.

$$\Rightarrow I\frac{d^2\theta}{dt^2}\underline{k} = (\underline{r_a} - z\underline{k}) \times \frac{T}{L}(\underline{r_D} - \underline{r_a}) + (\underline{r_b} - z\underline{k}) \times \frac{T}{L}(\underline{r_E} - \underline{r_b})$$
$$+ (\underline{r_c} - z\underline{k}) \times \frac{T}{L}(\underline{r_F} - \underline{r_c})$$

$$= \frac{T}{L}\Big[\underline{r_a} \times (\underline{r_D} - \underline{r_a}) - z\underline{k} \times (\underline{r_D} - \underline{r_a}) + \underline{r_b} \times (\underline{r_E} - \underline{r_b})$$
$$- z\underline{k} \times (\underline{r_E} - \underline{r_b}) + \underline{r_c} \times (\underline{r_F} - \underline{r_c}) - z\underline{k} \times (\underline{r_F} - \underline{r_c})\Big]$$

$$= \frac{T}{L}\Big[\underline{r_a} \times \underline{r_D} - \cancelto{0}{\underline{r_a} \times \underline{r_a}} - z\underline{k} \times \underline{r_D} + z\underline{k} \times \underline{r_a} + \underline{r_b} \times \underline{r_E}$$
$$- \cancelto{0}{\underline{r_b} \times \underline{r_b}} - z\underline{k} \times \underline{r_E} + z\underline{k} \times \underline{r_b} + \underline{r_c} \times \underline{r_F} - \cancelto{0}{\underline{r_c} \times \underline{r_c}}$$
$$- z\underline{k} \times \underline{r_F} + z\underline{k} \times \underline{r_c}\Big]$$

$$= \frac{T}{L}\Big[(\underline{r_a} \times \underline{r_D}) + (\underline{r_b} \times \underline{r_E}) + (\underline{r_c} \times \underline{r_F}) + z\underline{k} \times (\underline{r_a} + \underline{r_b} + \underline{r_c})$$
$$- z\underline{k} \times (\underline{r_D} + \underline{r_E} + \underline{r_F})\Big]$$

$$= \frac{T}{L}\Big[(\underline{r_a} \times \underline{r_D}) + (\underline{r_b} \times \underline{r_E}) + (\underline{r_c} \times \underline{r_F}) + \cancelto{0}{z\underline{k} \times 3z\underline{k}}$$
$$- \cancelto{0}{z\underline{k} \times 3L\underline{k}}\Big] \quad \text{——— } (*)$$

$$\underline{r_a} \times \underline{r_D} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ R\cos\theta & R\sin\theta & z \\ R & 0 & L \end{vmatrix} = LR\sin\theta\,\underline{i} - R(L\cos\theta - z)\,\underline{j}$$
$$+ R^2\sin\theta\,\underline{k}$$

Since the LHS of the above equation $(*)$ has only $\underline{k}$, we need to consider only the $\underline{k}$ component on RHS.

$$\underline{r}_b \times \underline{r}_E = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ R\cos\left(\frac{2\pi}{3}+\theta\right) & R\sin\left(\frac{2\pi}{3}+\theta\right) & z \\ R\cos\left(\frac{2\pi}{3}\right) & R\sin\left(\frac{2\pi}{3}\right) & L \end{vmatrix}$$

$$= \underline{i}\left(R^2\cos\left(\frac{2\pi}{3}+\theta\right)\sin\frac{2\pi}{3} - R^2\sin\left(\frac{2\pi}{3}+\theta\right)\cos\frac{2\pi}{3}\right)$$

$$- \underline{j}\left(LR\cos\left(\frac{2\pi}{3}+\theta\right) - Rz\cos\frac{2\pi}{3}\right) + \underbrace{\underline{k}\left(R^2\cos\left(\frac{2\pi}{3}+\theta\right)\sin\frac{2\pi}{3} - R^2\sin\left(\frac{2\pi}{3}+\theta\right)\cos\frac{2\pi}{3}\right)}_{-\underline{k}\,R^2\sin\theta}$$

(using the identity: $\sin(x+y) = \sin x\cos y + \cos x \sin y$)

Similarly

$$\underline{r}_c \times \underline{r}_F = \underline{i}\left[R^2\cos\left(\frac{4\pi}{3}+\theta\right)\sin\frac{4\pi}{3} - R^2\sin\left(\frac{4\pi}{3}+\theta\right)\cos\frac{4\pi}{3}\right]$$

$$- \underline{j}\left[LR\cos\left(\frac{4\pi}{3}+\theta\right) - Rz\cos\frac{4\pi}{3}\right] + \underbrace{\underline{k}\left[R^2\cos\left(\frac{4\pi}{3}+\theta\right)\sin\frac{4\pi}{3} - R^2\sin\left(\frac{4\pi}{3}+\theta\right)\cos\frac{4\pi}{3}\right]}_{-\underline{k}\,R^2\sin\theta}$$

You can verify that the sum of the $\underline{i}$ components and $\underline{j}$ components add upto zero.

$$\Rightarrow I\frac{d^2\theta}{dt^2} = \left(-3R^2\sin\theta\right)\frac{T}{L} \Rightarrow \boxed{I\frac{d^2\theta}{dt^2} + \frac{3R^2T}{L}\sin\theta = 0}$$

(j) For small values of θ, $\sin\theta \sim \theta$.

$$\Rightarrow \omega_n = \sqrt{\frac{3R^2T}{IL}} \text{ ———— } *$$

(k) Let's call the time period from (j) above, i.e.

the approximate time period $T_{app} = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{3R^2T}{IL}}}$ —(2)

Now, we can calculate the actual time period (T_{real}) for a given value of θ_0 by solving the eom (1) with MATLAB. To write MATLAB code, we need numerical values for R, T, I, L. We can proceed in one of 2 ways.

(i) Assume arbitrary values for $R, T, I, L, \ldots$

Get T_{app} from eq (2) above.

Solve EOM with MATLAB

$$\frac{d}{dt}\begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} \omega \\ -\frac{3R^2T}{IL}\sin\theta \end{bmatrix}$$, Initial conditions $[\theta_0, 0]$

Get T_{real} as a function of θ_0.

Find the range of θ_0 for which T_{app} & T_{real} are within 5% and 1% of each other.

(ii) Alternatively, we can eliminate the physical parameters by normalizing the time as

$\tau = \dfrac{t}{T_{app}}$. The the EOM becomes

$$\frac{1}{T_{app}^2}\frac{d^2\theta}{d\tau^2} + \frac{4\pi^2}{T_{app}^2}\sin\theta = 0 \Rightarrow \boxed{\frac{d^2\theta}{d\tau^2} + 4\pi^2 \sin\theta = 0}$$

This equation is now in term of non-dimensional quantities only. Linearize it: $\dfrac{d^2\theta}{d\tau^2} + 4\pi^2\theta = 0$

$\Rightarrow$ Approximate time period $\tau_{app} = 1$

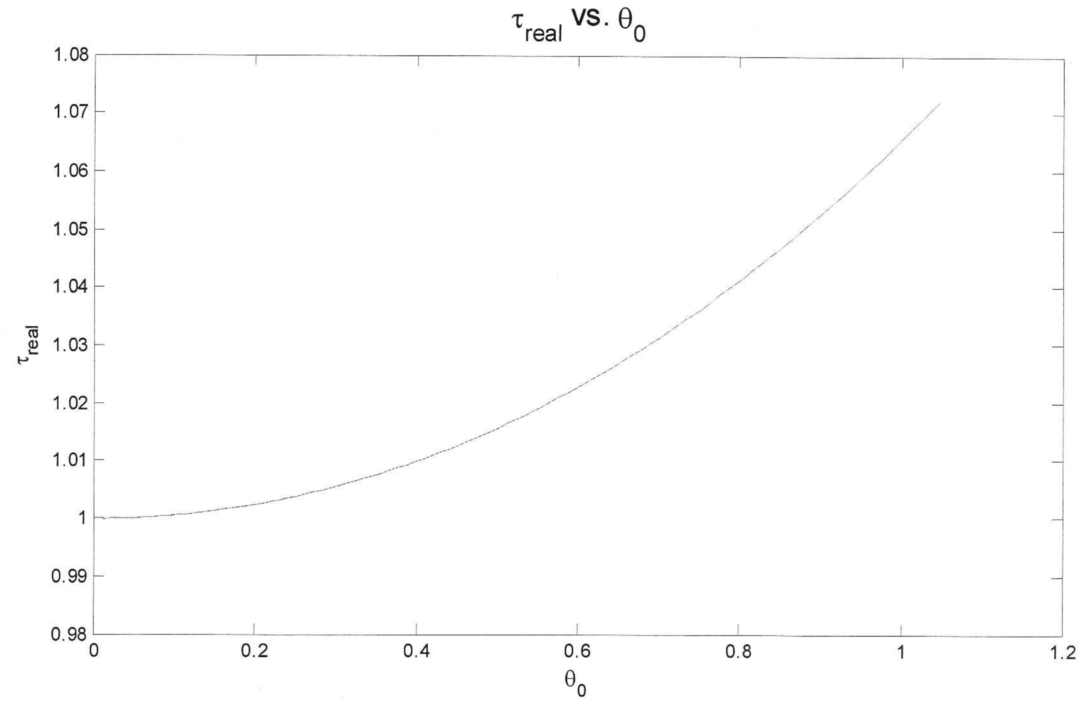
Use MATLAB to find τ_{real} as a function of θ_0.

Find the range of θ_0 for which τ_{app} & τ_{real} are within 5% & 1% of each other.

For 5% error, $\theta_0 = 0.8692$ radians

for 1% error, $\theta_0 = 0.3979$ radians.

```
function hw7problem3
    close all;

    % write a for loop to solve the equation of motion (eom) for
different
    % values of theta_0 between 0 and pi/3.
    for i=1:100
        theta0(i)=pi/3*i/100;
        ic=[theta0(i),0];
        options=odeset('Event',@detect_zero_crossing);
        [t,w,tevent,wevent,index]=ode45(@eom,[0,5],ic,options);
        % time difference between two successive events is t_real
        t_real(i)=tevent(3)-tevent(2);
    end

    plot(theta0,t_real)

    % following is the code to detect the range of theta_0 for which
    % t_real is within 5% and 1% of t_app

    for i=1:100
        if (t_real(i)<1.05)
            theta0_5=theta0(i);
        end
        if (t_real(i)<1.01)
            theta0_1=theta0(i);
        end
    end

    % write the values of theta_0 for 5% error adn 1% error to the
screen.
    theta0_5
    theta0_1

    % function of the eom in terms of the normalized time "tau"
    function dwdt=eom(t,w)
        dwdt=[w(2);-4*pi^2*sin(w(1))];
    end

    % event function to detect when theta crosses zero with positive
slope
    function [value,stop,dir]=detect_zero_crossing(t,w)
        value=w(1);
        stop=0;
        dir=1;
    end

end
```



τreal vs. θ0
1.08
1.07
1.06
1.05
1.04
1.03
1.02
1.01
1
0.99
0.98
τreal
0
0.2
0.4
0.6
0.8
1
1.2
θ0