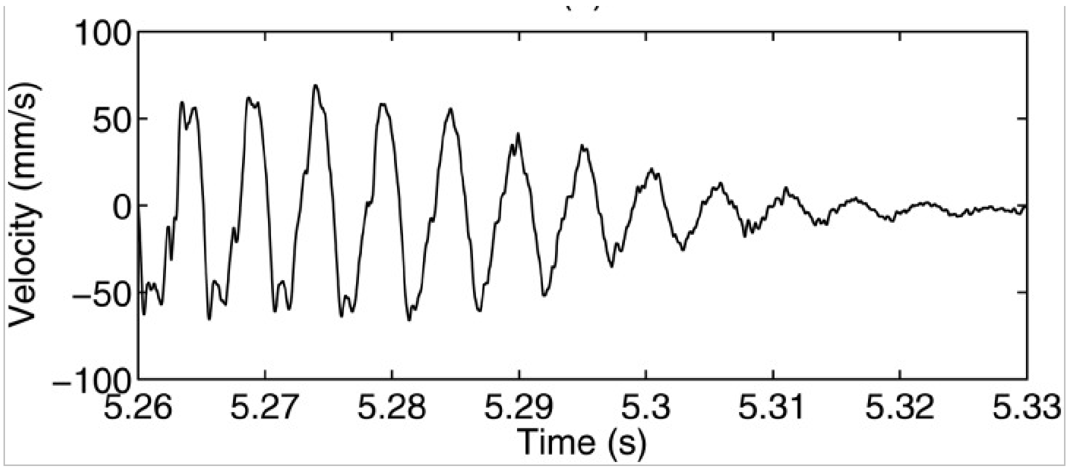
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 5: Free Vibrations - Solutions
Due Friday March 20th 2015
50 POINTS TOTAL



1. The figure shows a laser-vibrometry measurement of the velocity of a vocal-fold (from this paper). For the time interval 5.26s<t<5.29s, estimate

1.1 The amplitude and frequency of the vibration (give the frequency both in Hertz and in radians per second)

The amplitude is about 50 mm/s

There are 4 cycles between 5.26s and 5.28s, so the period is 0.02/4=0.005s. The frequency is 200 Hz, or 1257 rad/s

[2 POINTS]

1.2 The amplitude of the displacement.

For harmonic vibrations $x = X_0 \sin \omega t \qquad v = \omega X_0 \cos \omega t \qquad a = -\omega^2 X_0 \sin \omega t$ so

$X_0 = 50/1257 = 0.0398mm = 39.8\mu m$

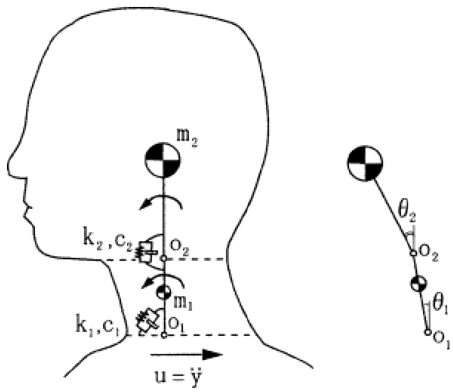
[1 POINT]

1.3 The amplitude of the acceleration.

Similarly $A_0 = \omega V_0 = 1257 \times 50mm/s^2 = 62.8m/s^2$

[1 POINT]

2. State the number of degrees of freedom and the number of natural frequencies of vibration for each of the systems shown below



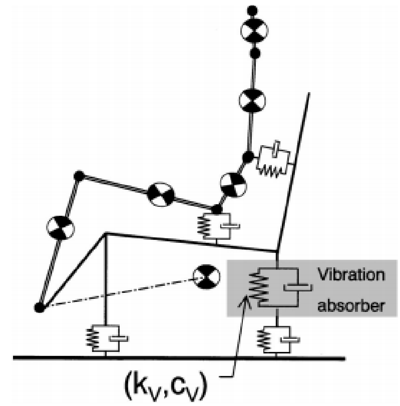


Fig. 6 Simulation model with vibration absorber

(a) Model of a human head/neck

(b) 2D Model of a patient on a wheelchair.
(The seat can only move vertically, and each link in the chain is a rigid body that is connected to its neighbors by a pin joint)

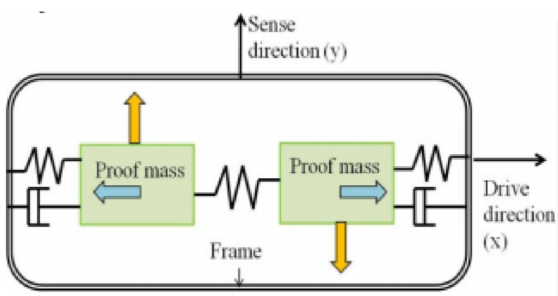
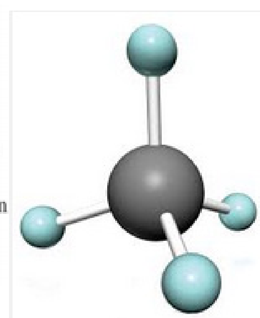


Figure 3: Two Proof Mass-Spring-Damper System



(c) Model of a MEMS gyroscope (the masses are particles, and move in the *x,y* plane)

(d) Methane molecule (the balls are particles, the rods are springs)

(a) The figure shows that the system can be described by two angles, so 2 DOF. There are no rigid body modes, so 2 natural frequencies.
(b) We need to specify the height of the seat, the orientation of the seat, and the angles between the rigid links at 5 joints, so7 in total. Or, from the formula, there are 6 rigid bodies, and 11 constraints (one for the seat, and two at each pin joint), giving 18-11=7DOF. No rigid body modes, 7 nat. freqs.
(c) 2 particles, no constraints, and 2D, 4 DOF. No rigid body modes, so 4 natural frequencies.
(d) 5 particles, so 15 DOF. There are 6 rigid body modes, so 9 natural frequencies.

[4 POINTS TOTAL]

3. Solve the following differential equations (use the Solutions to Differential Equations)

3.1 $\frac{d^2 y}{dt^2} + 4y = 0 \qquad y = 1 \quad \frac{dy}{dt} = 0 \quad t = 0$

The formula sheet gives the solution to $\frac{1}{\omega_n^2}\frac{d^2 x}{dt^2} + x = C$ with $x = x_0 \qquad \frac{dx}{dt} = v_0 \qquad t = 0$ is

$x(t) = C + (x_0 - C)\cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t$

Here $C = 0 \ \ \omega_n = 2, \ x_0 = 1, \ \ v_0 = 0$ so $x(t) = \cos 2t$

[2 POINTS]

3.2 $\frac{d^2 y}{dt^2} + 4\frac{dy}{dt} + 16y = 16\sin(4t) \qquad y = 0 \quad \frac{dy}{dt} = 0 \quad t = 0$

The equation in standard form is $\frac{1}{4^2}\frac{d^2 y}{dt^2} + \frac{1}{4}\frac{dy}{dt} + y = \sin(4t) \qquad y = 0 \quad \frac{dy}{dt} = 0 \quad t = 0$

so $\zeta = 1/2 \ \ \omega_n = 4 \ \ \omega = 4 \ \ KF_0 = 1$

The formula gives $x(t) = C + x_h(t) + x_p(t)$ with steady state solution

$$x_p(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = \frac{KF_0}{\left\{\left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} = 1 \qquad \phi = \tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1 - \omega^2/\omega_n^2} = -\frac{\pi}{2}$$

The system is underdamped, so the transient solution is

$$x_h(t) = \exp(-\varsigma\omega_n t)\left\{x_0^h \cos\omega_d t + \frac{v_0^h + \varsigma\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$$

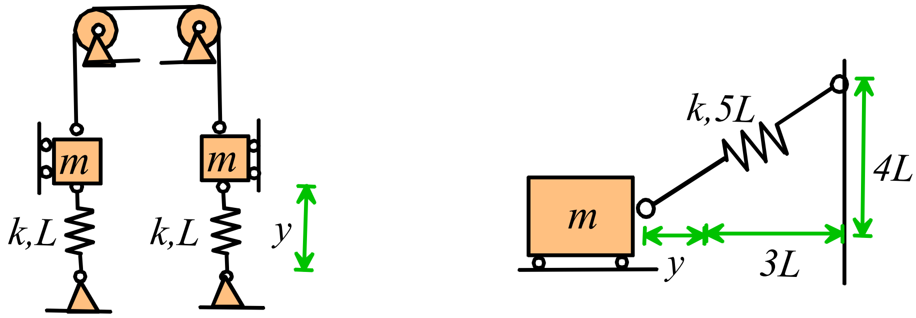
where $\omega_d = \omega_n\sqrt{1-\varsigma^2} = 4\sqrt{1 - 1/4} = 2\sqrt{3}$

$$x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0\sin\phi = 1$$

$$v_0^h = v_0 - \left.\frac{dx_p}{dt}\right|_{t=0} = v_0 - X_0\omega\cos\phi = 0$$

Thus $x(t) = \sin\left(4t - \frac{\pi}{2}\right) + \exp(-2t)\left\{\cos 2\sqrt{3}t + \frac{1}{\sqrt{3}}\sin 2\sqrt{3}t\right\}$ (of course $\sin(4t - \pi/2) = -\cos(4t)$)

[3 POINTS]



4. For the two conservative single-degree of freedom systems shown in the figure:

4.1 Derive the equation of motion (use energy methods, and include gravity. The pulleys and cable are massless). State whether the equation of motion is linear or nonlinear.

The potential energies for the two systems are:

(a) $V = k(y-L)^2$ (no gravity because when one mass moves up the other moves down)

(b) $V = \frac{k}{2}\left(\sqrt{(3L+y)^2+(4L)^2} - 5L\right)^2 = \frac{k}{2}\left(\sqrt{(3L+y)^2+(4L)^2} - 5L\right)^2$

The kinetic energies are

(a) $m\left(\frac{dy}{dt}\right)^2$ (b) $\frac{1}{2}m\left(\frac{dy}{dt}\right)^2$

The systems are conservative so $\frac{d}{dt}(T+V)=0$

Evaluating the derivatives gives

(a) $2m\frac{d^2y}{dt^2}\left(\frac{dy}{dt}\right) + 2k(y-L)\left(\frac{dy}{dt}\right) = 0 \Rightarrow \frac{m}{k}\frac{d^2y}{dt^2} + y = L$

$$m\frac{d^2y}{dt^2}\left(\frac{dy}{dt}\right) + k\left(\sqrt{(3L+y)^2+(4L)^2} - 5L\right)\frac{3L+y}{\sqrt{(3L+y)^2+(4L)^2}}\left(\frac{dy}{dt}\right) = 0$$

(b)

$$\Rightarrow \frac{m}{k}\frac{d^2y}{dt^2} + \left(1 - \frac{5L}{\sqrt{(3L+y)^2+(4L)^2}}\right)(3L+y) = 0$$

The first system is linear, the second is nonlinear.

[4 POINTS]

4.2 If appropriate, linearize the equation of motion for small amplitude vibrations (that means doing that Taylor series stuff discussed in class. "Linearizing" means replacing the nonlinear function of the variable with an approximate linear function)

The first system is already linear, so no need to rearrange. For the second we need to linearize

$$\left(1-\frac{5L}{\sqrt{(3L+y)^2+(4L)^2}}\right)(3L+y).$$

Note that the Taylor expansion of the first term is $\left(1-\frac{5L}{\sqrt{(3L+y)^2+(4L)^2}}\right)\approx 0+\frac{15L^2}{(5L)^3}y+...$

Hence the linearized equation is $\frac{m}{k}\frac{d^2y}{dt^2}+\frac{9}{25}y=0$

You can also do the series in Mupad

```
assume(L>0)
```

```
(1-5*L/sqrt( (3*L+y)^2 + (4*L)^2 ))*(3*L+y)
```

$$-(3L+y)\left(\frac{5L}{\sqrt{16L^2+(3L+y)^2}}-1\right)$$

```
simplify(series(%,y,1))
```

$$\frac{9y}{25}+O(y^2)$$

[2 POINTS]

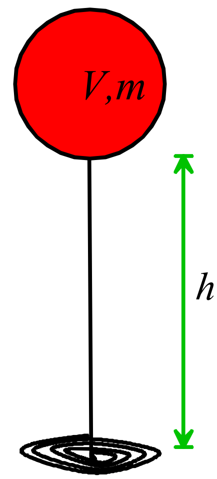
4.3 Arrange the (linearized) equation of motion into standard form, and find an expression for the natural frequency of vibration.

The first is already in standard form so $\frac{m}{k}\frac{d^2y}{dt^2}+y=0 \Rightarrow \omega_n=\sqrt{\frac{k}{m}}$

The second is $\frac{25m}{9k}\frac{d^2y}{dt^2}+y=0 \Rightarrow \omega_n=\frac{3}{5}\sqrt{\frac{k}{m}}$

[2 POINTS]

5. A helium balloon with total mass m and volume V is supported by a tether with mass per unit length μ. Part of the cable is coiled on the ground.



5.1 Assuming vertical motion, write down the total potential and kinetic energy of the balloon and cable together as a function of its height h. (The buoyancy force acting on the balloon is $g\rho V$ where ρ is the mass density of air. Since the balloon floats $mg < \rho V g$. Assume that the air current surrounding a balloon moving with velocity **v** has kinetic energy $\rho V|\mathbf{v}|^2/4$ - this comes from a [fluid mechanics calculation] of flow past a moving sphere)

The potential energy is gravity + buoyancy, which gives

$$V = mgh - \rho gVh + \frac{1}{2}\mu gh^2$$

where we have noted that the center of mass of the cable is half way up.

The kinetic energy is $T = \frac{1}{2}m\left(\frac{dh}{dt}\right)^2 + \frac{1}{2}\mu h\left(\frac{dh}{dt}\right)^2 + \frac{1}{4}\rho V\left(\frac{dh}{dt}\right)^2$

[2 POINTS]

5.2 Hence, find the equation of motion for h

$\frac{d}{dt}(T+V) = 0$ gives

$$\left(m + \frac{1}{2}\rho V + \mu h\right)\frac{d^2h}{dt^2}\left(\frac{dh}{dt}\right) + \frac{1}{2}\mu\left(\frac{dh}{dt}\right)^3 + \left(mg - \rho gV + \mu h\right)\left(\frac{dh}{dt}\right) = 0$$

$$\Rightarrow\left(m + \frac{1}{2}\rho V + \mu h\right)\frac{d^2h}{dt^2} + \frac{1}{2}\mu\left(\frac{dh}{dt}\right)^2 + \left(mg - \rho gV + \mu hg\right) = 0$$

[2 POINTS]

5.3 Find the value of h for which the balloon is in static equilibrium.

The acceleration and velocity are zero in equilibrium so the equation of motion gives

$$h = \frac{(\rho V - m)}{\mu}$$

[1 POINT]

5.4 Assuming that the balloon is disturbed by a small distance δx from the equilibrium height found in 5.3, linearize the equation and hence find the natural frequency (to linearize you will need to assume that displacement, velocity and acceleration are all sufficiently small to neglect quadratic or higher order terms. Products of δx and acceleration can also be neglected).

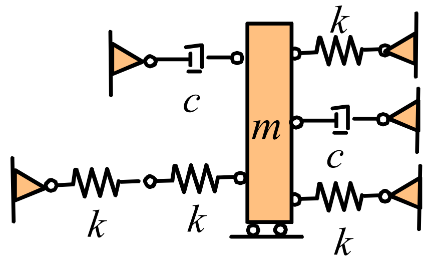
We set $h = \frac{(\rho V - m)}{\mu} + \delta x$ in the equation of motion, and discard high order terms, to get

$\left(\frac{3}{2}\rho V\right)\frac{d^2\delta x}{dt^2} + \mu g\delta x = 0 \Rightarrow \left(\frac{3\rho V}{2\mu g}\right)\frac{d^2\delta x}{dt^2} + \delta x = 0$. The natural frequency follows as $\omega_n = \sqrt{\left(\frac{2\mu g}{3\rho V}\right)}$

[2 POINTS]

Notice that the frequency is independent of the balloon mass….

6. Replace the system shown in the figure with an equivalent spring-mass system consisting of a mass with only one spring and dashpot. Hence, determine a formula for the undamped natural frequency and the damping factor for the system.



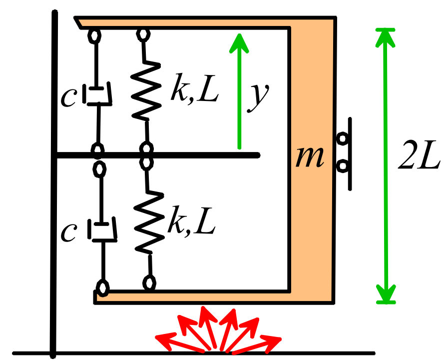
The two springs connected end to end are in series, and have a stiffness

$\frac{1}{k_{eff}} = \frac{1}{k} + \frac{1}{k} \Rightarrow k_{eff} = \frac{k}{2}$. Everything else is in parallel, and stiffnesses or dashpot coefficients just add. The total stiffness is thus $5k/2$ and the total dashpot coefficient is $2c$.

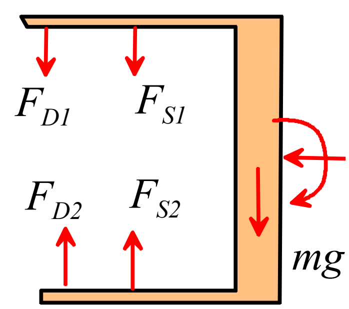
The natural frequency follows as $\omega_n = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{5k}{2m}}$ $\qquad \zeta = \frac{c_{eff}}{2\sqrt{k_{eff} m}} = \frac{c}{\sqrt{5km/2}}$

[2 POINTS]

7. The figure (from this paper) shows an instrument that is designed to measure the vertical impulse exerted by the explosion of a buried charge. A mass m is supported by springs and dampers attached to a rigid frame. The vertical displacement of the mass is measured after the explosion.



7.1 Derive the equation of motion for the length y.



A free body diagram is shown. We know that

$$F_{S1} = k(y-L) \qquad F_{S2} = k(2L-y-L)$$

$$F_{D1} = c\frac{dy}{dt} \qquad F_{D2} = -c\frac{dy}{dt}$$

F=ma in the vertical direction gives

$$m\frac{d^2y}{dt^2} = -2c\frac{dy}{dt} - 2k(y-L) \Rightarrow \frac{m}{2k}\frac{d^2y}{dt^2} + \frac{c}{k}\frac{dy}{dt} + y = L - \frac{mg}{2k}$$

[2 POINTS]

7.2 Calculate a formula for y when the system is in static equilibrium (before the charge is fired).

In equilibrium all the time derivatives vanish, so $y = L - \frac{mg}{2k}$

[1 POINT]

7.3The system starts at rest with y at its equilibrium position. The explosive charge exerts an impulse I on the mass m. The system is designed to be under-damped. Write down the velocity of the mass just after the explosion, and hence show that

$$y(t) = L - \frac{mg}{2k} + \frac{I}{\sqrt{2km(1-\zeta^2)}} \exp(-\zeta\omega_n t)\sin\omega_d t$$

and give formulas for $\omega_n, \zeta\ \omega_d$,

Momentum conservation gives $I = mv_0$

The equation of motion can be expressed in standard form by setting $\omega_n = \sqrt{\frac{2k}{m}} \qquad \zeta = \frac{2c}{2\sqrt{2km}}$

We now can solve the equation of motion – this is a Case III free vibration problem with solution

$$x(t) = C + \exp(-\varsigma\omega_n t)\left\{(x_0 - C)\cos\omega_d t + \frac{v_0 + \varsigma\omega_n(x_0 - C)}{\omega_d}\sin\omega_d t\right\}$$

Here $x_0 = C$, $v_0 = I/m$ $\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}(1-\zeta^2)}$. Substituting into the solution and simplifying gives the answer stated.

[3 POINTS]

7.4 The actual design has the following parameters:

- k=20 MN/m
- c=50 kNs/m
- m=25000 kg

The maximum displacement of the frame (measured relative to its static equilibrium position) was found to be 4.63 cm. Find the values of ω_n, ζ for the instrument and hence determine the magnitude of the impulse.

$$\omega_n = \sqrt{\frac{2k}{m}} = 40 rad/s \qquad \zeta = \frac{2c}{2\sqrt{2km}} = 0.05$$

The maximum value of $\exp(-\zeta\omega_n t)\sin\omega_d t$ can be found with calculus – differentiating and setting to zero to find the location of the maximum

$$\exp(-\zeta\omega_n t)\left(-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t\right) = 0$$

$$\Rightarrow -\sqrt{(\zeta\omega_n)^2 + \omega_d^2}\left(\cos\psi \sin\omega_d t - \sin\psi \cos\omega_d t\right) = 0 \Rightarrow \sin(\omega_d t - \psi) = 0 \qquad \psi = \sin^{-1}\frac{\omega_d}{\sqrt{(\zeta\omega_n)^2 + \omega_d^2}}$$

$$\Rightarrow \omega_d t = \psi$$

and hence $\exp(-\zeta\omega_n\psi)\sin\omega_d t = \sqrt{1-\zeta^2}\exp(-\zeta\omega_n \sin^{-1}\sqrt{1-\zeta^2})$. The maximum deflection follows as

$$\frac{I}{\sqrt{2km}}\exp\left(-\frac{\zeta\omega_n}{\omega_d}\sin^{-1}\sqrt{1-\zeta^2}\right) \approx \frac{I}{\sqrt{2km}}\exp(-\zeta\pi/2) = \frac{I}{10^6}\exp(-0.05\pi/2)$$

$$\Rightarrow \mathrm{I} = 0.0463 \times \exp(0.05\pi/2) \times 10^6 = 50kNs$$

You can also do the maximization with mupad using numbers

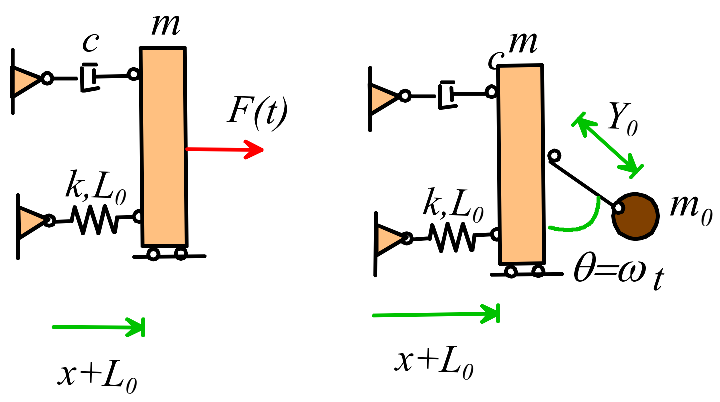
```
k := 20000000.0:
c := 50000:
m := 25000:
wn := sqrt(2*k/m):
z := 2*c/(2*sqrt(2*k*m)):
wd := wn*sqrt(1-z^2)
 39.94996871
f := exp(-z*wn*t)*sin(wd*t):
solve(diff(f,t),t,IgnoreSpecialCases)[0]:
subs(f,t=%)
 e^-0.07613400055 sin(1.52077547)
float(%)
 0.9255329311
II := 0.0463*sqrt(2*k*m*(1-z^2))/%
 49962.66176
```

[3 POINTS]

Graders – numbers in this problem are fiddly – as long as the method is correct and the procedure is explained clearly deduct ½ point for getting the wrong numbers…

8. Determine the steady-state amplitude of vibration for the spring-mass systems shown in the figure (you don't need to derive the equations of motion – these are standard textbook systems and you can just use the standard formulas). In each case the mass *m=20*kg, the stiffness *k=2000N/m* *c=20Ns/m*.

The force is $F(t) = 20\sin 20t$ N ;, the length of the rotor is 5cm; the eccentric mass $m_0 = 4kg$ and the angular velocity of the rotor is $\omega = 40$ rad/s.



For the forced system the natural frequency and damping factor are $\omega_n = \sqrt{\dfrac{k}{m}} = 10 \qquad \zeta = \dfrac{c}{2\sqrt{km}} = 0.05$

The amplitude of vibration of the forced spring-mass system is

$$X_0 = \frac{F_0}{k}\frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} = \frac{20}{2000}\frac{1}{\sqrt{(1-20^2/100)^2 + (0.2)^2}} = 3.3mm$$

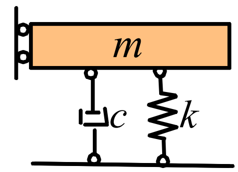
For the rotor excited system $\omega_n = \sqrt{\dfrac{k}{m+m_0}} = 9.13 \qquad \zeta = \dfrac{c}{2\sqrt{km}} = 0.0456$

The amplitude is

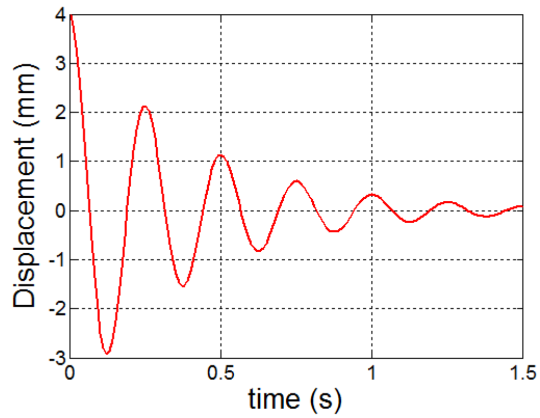
$$X_0 = \frac{Y_0 m_0}{m+m_0}\frac{\omega^2/\omega_n^2}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$$
$$= \frac{0.05\times 4}{24}\frac{40^2/9.13^2}{\sqrt{(1-40^2/9.13^2)^2 + (2\times 0.0456\times 40/9.13)^2}} = 8.6mm$$

[4 POINTS]

9. A vibration isolation platform can be idealized as a spring-mass-damper system as shown in the figure. In a free vibration test on the table, the base is held fixed and the platform is disturbed slightly from its equilibrium position. The subsequent displacement of the table is plotted in the figure below as a function of time.



9.1 Use the graph provided to estimate the period of oscillation and the log decrement. Hence, calculate the natural frequency ω_n and damping factor ζ that characterize the vibration isolation table.



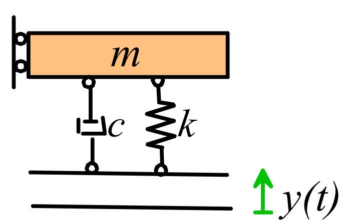
There are two cycles in 0.5 sec so the period is 0.25 sec

Using the first and 3rd peaks the log decrement is log(4)/2 (use the natural log) is 0.693

The standard formulas then give $\zeta = \dfrac{\delta}{\sqrt{4\pi^2 + \delta^2}} \approx 0.1$ $\qquad \omega_n = \dfrac{\sqrt{4\pi^2 + \delta^2}}{T} \approx 25$ rad/s

[2 POINTS]

9.2 The base of the platform is subjected to a harmonic displacement $y(t) = Y_0 \sin \omega t$ with amplitude 5mm and frequency $(25/\pi)$ Hz. Calculate the amplitude of vibration of the platform.



The angular frequency of the base is 50 rad/s.

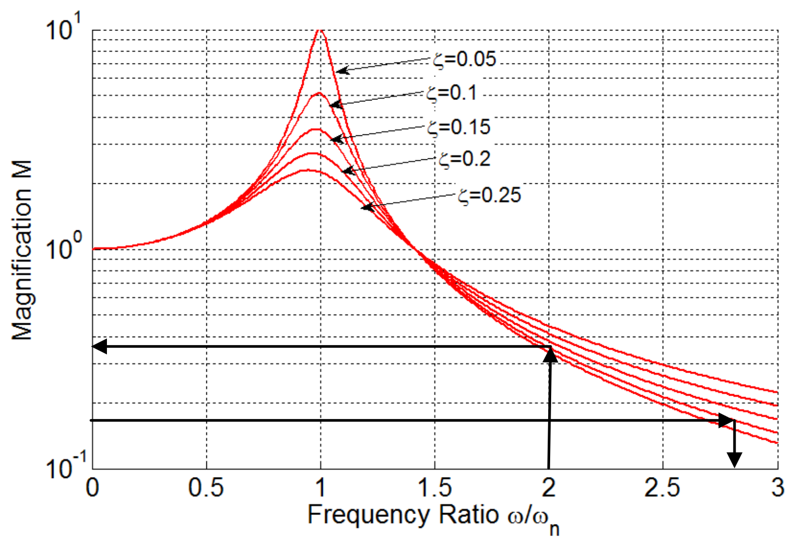
The displacement amplitude follows from the standard formula

$$Y_0 \frac{\sqrt{1+\left(\dfrac{2\zeta\omega}{\omega_n}\right)^2}}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2+\left(\dfrac{2\zeta\omega}{\omega_n}\right)^2}} = 5\frac{\sqrt{1+\left(\dfrac{2\times 50}{10\times 25}\right)^2}}{\sqrt{\left(1-\dfrac{50^2}{25^2}\right)^2+\left(\dfrac{2\times 50}{10\times 25}\right)^2}} = 1.8mm$$

[2 POINTS]

9.3 It is necessary to modify the vibration isolation system to further reduce the vibration amplitude by a factor of two. Recommend changes to the values of *k,m*, and/or *c* necessary to achieve this (e.g. recommend that *k* should be increased by some factor, *m* should be reduced by some factor, etc).

The figure shows the plot of magnification –v- frequency for a base-excited system.



In the original design $\omega/\omega_n = 2,\ \zeta = 0.1$ and the magnification $M = 1.8/5 = 0.36$.

It is clear that changing only the damping won't be enough to achieve the necessary reduction in amplitude. A better approach is to reduce the natural frequency. We can do this graphically – we need to reduce the magnification to 0.36/2=0.18, which requires $\omega/\omega_n \approx 2.75$.

Alternatively we can solve

$$5\frac{\sqrt{1+\left(\frac{2\times 50}{10\times\omega_n}\right)^2}}{\sqrt{\left(1-\frac{50^2}{\omega_n^2}\right)^2+\left(\frac{2\times 50}{10\times\omega_n}\right)^2}}=1.8/2$$

```
sqrt(1+(10/ww)^2)/sqrt( (1-50^2/ww^2)^2 + (10/ww)^2) = 1.8/2/5
```

$$\frac{\sqrt{\frac{100}{ww^2}+1}}{\sqrt{\frac{100}{ww^2}+\left(\frac{2500}{ww^2}-1\right)^2}}=0.18$$

```
solve(%,ww)
```

$$\{-18.51757069,\ 18.51757069,\ 24.70475622\,i,\ -24.70475622\,i\}$$

Either way we need to reduce ω_n below 18 rad/s. We thus need to scale $\sqrt{\frac{k}{m}}$ by 18/25. This can be achieved by halving the spring stiffness, or doubling the mass, or if you want to leave the damping unchanged, increase the mass by $\sqrt{2}$ and reduce the spring stiffness by $1/\sqrt{2}$

Graders – numbers will vary depending on how the graph is read – and any sensible change in k and m is fine, it doesn't have to be exactly a factor of 2.

[3 POINTS]