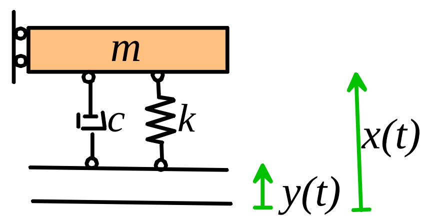
School of Engineering
Brown University

EN40: Dynamics and Vibrations

Homework 6: Forced Vibrations
Due Friday April 6, 2018

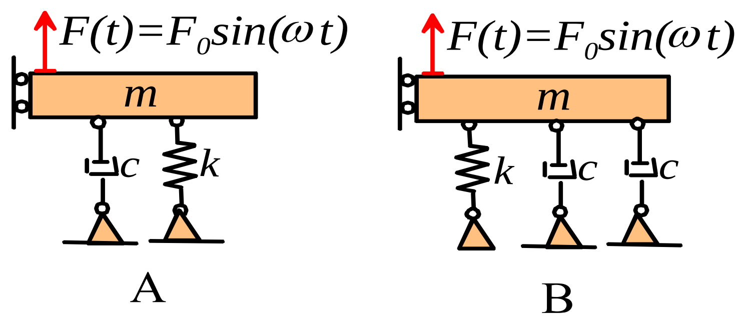
1. The vibration isolation system shown in the figure has

- m=20kg,
- $k = 19.8kN/m$
- $c = 1.259kNs/m$



If the base vibrates harmonically with an amplitude of 1mm and frequency of 100Hz, what is the steady-state amplitude of vibration of the platform (i.e. the mass m)?

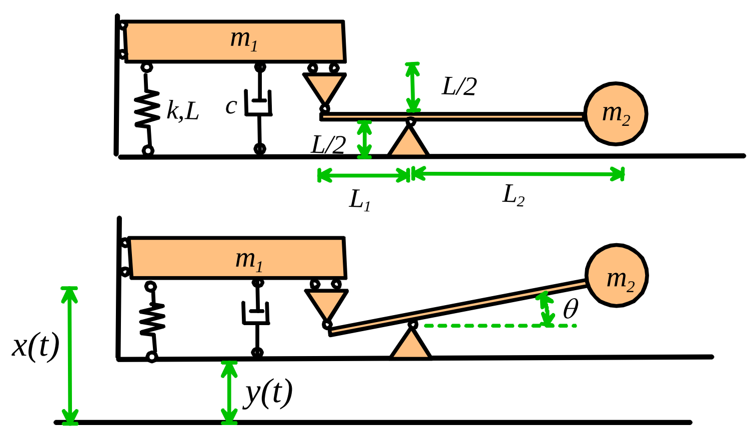
2. Both systems in the figure are subjected to a force with amplitude 1 kN and frequency equal to the undamped natural frequency of the spring-mass system ($\omega = \omega_n$).



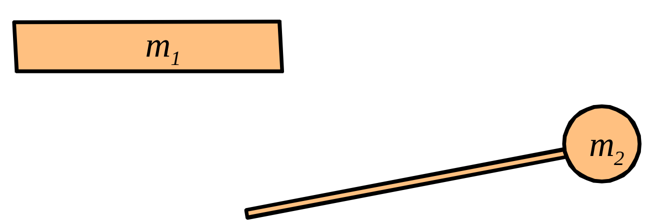
The vibration amplitude of system B is measured to be 1mm.

What is the vibration amplitude of system A?

3. In this (hard!) problem we will analyze the behavior of the 'anti-resonant' vibration isolation system introduced in Homework 5. The system is illustrated in the figure. Assume that the base vibrates vertically with a displacement $y(t) = Y_0 \sin \omega t$. Our goal is to calculate a formula for the steady-state vertical motion $x(t)$ of the platform, and to compare the behavior of this system with the standard base excited spring-mass-damper design for an isolation system.



3.1 Draw free body diagrams showing the forces acting on the mass m_1 and the pendulum assembly (see the figure).



3.2 Using geometry, find an expression for the acceleration of mass m_2 in terms of θ , y and their time derivatives (as well as relevant geometric constants) (e.g. by writing down a formula for the position vector relative to a fixed origin and differentiating it). Show that if θ and its time derivatives are small the result can be approximated by

$$\mathbf{a} \approx \left[\frac{d^2 y}{dt^2} + L_2 \frac{d^2 \theta}{dt^2} \right] \mathbf{j}$$

Show also that if θ and its time derivatives are small

$$\frac{d^2 x}{dt^2} - \frac{d^2 y}{dt^2} \approx -L_1 \frac{d^2 \theta}{dt^2}$$

3.3 For the pendulum, write down $\mathbf{F} = m\mathbf{a}$ and $\mathbf{M} = 0$ about the center of mass, in terms of reaction forces shown in your FBD. Use the approximation in 3.2 for the acceleration.

3.4 Write down $\mathbf{F} = m\mathbf{a}$ for mass m_1 , and hence use 3.3 and the second of 3.2 to show that (if if θ and its time derivatives are small) then

$$\left(m_1 + m_2 \frac{L_2^2}{L_1^2} \right) \frac{d^2 x}{dt^2} + c\frac{dx}{dt} + kx = m_2 \frac{L_2}{L_1} \left(1 + \frac{L_2}{L_1} \right) \frac{d^2 y}{dt^2} + c\frac{dy}{dt} + ky + kL$$

3.5 Show that the equation can be re-arranged into the form

$$\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = \frac{\lambda^2}{\omega_n^2} \frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy}{dt} + y + C$$

with

$$\omega_n = \sqrt{\left(\frac{kL_1^2}{L_1^2 m_1 + m_2 L_2^2} \right)} \qquad \lambda = \sqrt{\left(\frac{m_2 L_2 \left(L_1 + L_2 \right)}{\left(L_1^2 m_1 + m_2 L_2^2 \right)} \right)}$$

$$\zeta = \frac{c}{2\sqrt{k(m_1 L_1^2 + m_2 L_2^2) / L_1^2}}$$

(and figure out what C is!)

3.6 Suppose that the base is subjected to harmonic excitation $y = Y_0 \sin \omega t$. Show (using calculus and the double-angle formula $\cos\psi \sin \omega t + \sin\psi \cos \omega t = \sin(\omega t + \psi)$) that the right hand side of the differential equation can be re-written as

$$\frac{\lambda^2}{\omega_n^2}\frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = Y_0\left\{\left(1 - \frac{\lambda^2 \omega^2}{\omega_n^2}\right)\sin \omega t + \frac{2\zeta\omega}{\omega_n}\cos \omega t\right\}$$

$$= Y_0\sqrt{\left(1 - \frac{\lambda^2\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}\sin(\omega t + \psi) \quad \psi = \tan^{-1}\frac{2\zeta\omega / \omega_n}{1 - \lambda^2(\omega / \omega_n)^2}$$

Hence use the 'Case IV' solution to differential equations (just make F_0 a suitable function of frequency) to show that the steady state solution for x has the form

$$x(t) = L + X_0 \sin(\omega t + \phi) \qquad X_0 = M(\omega / \omega_n, \zeta, \lambda)Y_0$$

and show that the magnification factor M is given by

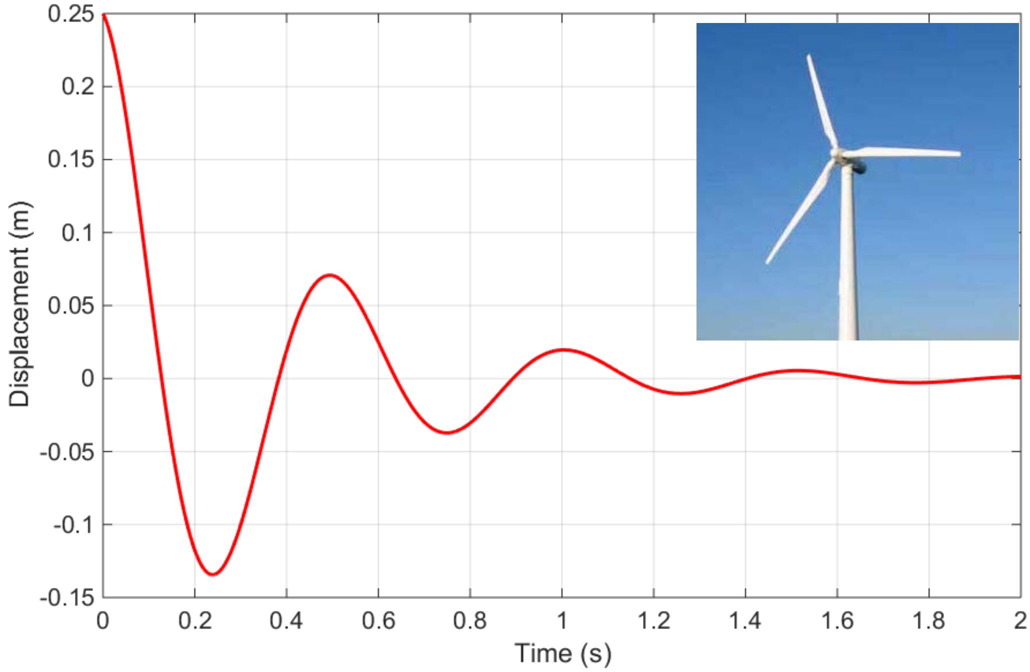
$$M = \frac{\sqrt{\left(1 - \frac{\lambda^2\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}$$

3.7. Plot a graph of M as a function of $0 < \omega / \omega_n < 6$ for $\lambda = 0$,for values of $\zeta = 0.02, 0.05, 0.1, 0.2$ (on the same plot). This graph shows the magnification for the 'standard' base excited vibration isolation system discussed in class, since $\lambda = 0$ corresponds to a pendulum with zero mass – it should look the same as the 'Case V' magnification graph. For comparison, plot a second graph of M as a function of ω / ω_n for $\lambda = 0.6$ (an anti-resonant isolator),for values of $\zeta = 0.02, 0.05, 0.1, 0.2$.

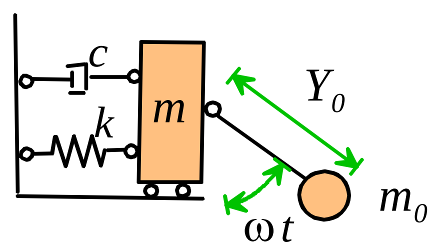
3.8 What is the frequency corresponding to the anti-resonance (the minimum value of M), in terms of λ, ω_n (give an approximate solution for $\zeta << 1$) ? What is (approximately) the smallest vibration amplitude (in terms of λ, ζ)?

3.9 For what range of frequency (in terms of λ , ω_n) does the pendulum system give better performance than the simpler spring-mass-damper system?

3.10 What sort of application would be best suited for an anti-resonant vibration isolator?



4 An unbalanced wind-turbine is idealized as a rotor-excited spring-mass system as shown in the figure. The mass m represents the tower, and m_0 represents the combined mass of the three rotor blades. The spring and damper represent the stiffness and energy dissipation in the tower. The rotor is 'unbalanced' because its center of mass is a small distance Y_0 away from the axle. The total mass $(m+m_0)$ of the system is 25000kg.



The figure shows the results of a free vibration experiment on the turbine.

4.1 Use the data provided to determine the following quantities:

(a) The vibration period

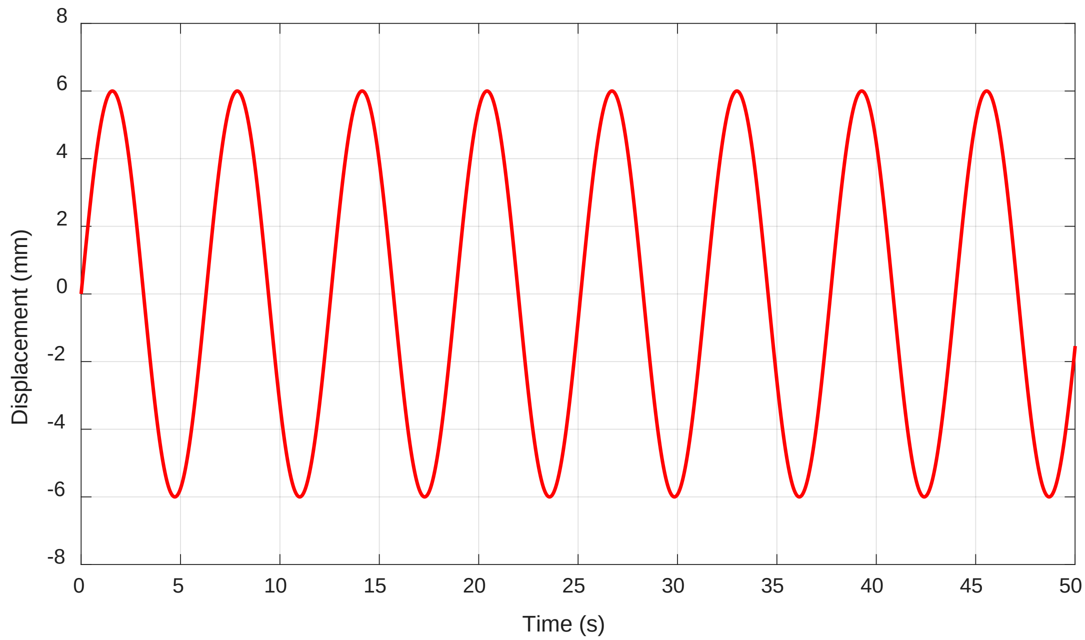
(b) The log decrement

(c) The undamped natural frequency

(d) The damping factor

(e) The spring stiffness

(f) The dashpot coefficient



4.2 The figure shows the measured displacement of the system during operation. The blades have a radius of 40m, and the total mass of the system $(m + m_0)$ is 25000kg . Assuming that the rotor can be balanced by adding mass to the tip of one blade, estimate the mass that must be added to balance the rotor.