Episode 2

Motion of Particles in Cartesian Coordinates

ENGN0040: Dynamics and Vibrations
Jesse Ault, Allan Bower, Yue Qi

School of Engineering
Brown University
Topics for today's class

Describing motion of particles

1. Definition of a particle
2. Position-velocity-acceleration relations for a particle
3. Inertial reference frames
4. Review of some calculus
5. Analyzing straight line motion of particles
6. Using MATLAB to integrate/differentiate the measured position/velocity/acceleration of a particle
Describing motion of particles

2.1 Definition

A particle is a point mass at some position in space.

Properties:
(a) Mass
(b) Position (velocity, acceleration)
No shape or orientation

Examples:
Satellite, in space (for orbit)
Atom in an MD simulation
2 Describing motion of particles

2.1 Definition A particle is a point mass at some position in space

Properties:
- Mass
- Position (Velocity, accel)
- No shape or orientation

Examples:
- Satellite in space (for orbit)
- Atom in an MD simulation
2.2 Position - Velocity - Acceleration formulas (Cartesian coords)

2.2.1 Position

\[ \mathbf{r} = x(t) \mathbf{i} + y(t) \mathbf{j} + z(t) \mathbf{k} \]

2.2.2 The "Inertial Basis"

To use Newton:

(a) \( \mathbf{O} \) must not accelerate
(b) \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) must not rotate

This is an approximation! Use judgement!
2.2.3 Velocity

\[ \mathbf{V} = V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k} \]

Speed \[ V = \sqrt{V_x^2 + V_y^2 + V_z^2} \]

Direction is tangent to path

Definition \[ \mathbf{V} = \frac{d\mathbf{r}}{dt} \]

\[ = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k} \]

Hence \[ V_x = \frac{dx}{dt} \quad V_y = \frac{dy}{dt} \quad V_z = \frac{dz}{dt} \]
2.2.4 Acceleration

\[ a = a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \]

**Definition**

\[
\begin{align*}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{dV_x}{dt} \mathbf{i} + \frac{dV_y}{dt} \mathbf{j} + \frac{dV_z}{dt} \mathbf{k} \\
&= \frac{d^2x}{dt^2} \mathbf{i} + \frac{d^2y}{dt^2} \mathbf{j} + \frac{d^2z}{dt^2} \mathbf{k}
\end{align*}
\]

Hence

\[
\begin{align*}
a_x &= \frac{dV_x}{dt} = \frac{d^2x}{dt^2} \\
a_y &= \frac{dV_y}{dt} = \frac{d^2y}{dt^2} \\
a_z &= \frac{dV_z}{dt} = \frac{d^2z}{dt^2}
\end{align*}
\]
2.2.5 Special Case: 1-D motion

\[ V = \frac{dx}{dt} \quad a = \frac{dv}{dt} = \frac{d^2x}{dt^2} \]

Formula for \( a \) in terms of \( x \)

Suppose we know \( V \) as a function of \( x \) (e.g. \( V = x^2 \))

Calculate \( a \) using chain rule

\[
a = \frac{dv}{dx} \frac{dx}{dt} = V \frac{dv}{dx}
\]

(For \( V = x^2 \) \( a = 2x \), \( V = 2x^3 \))
Particle motion problems we need to solve

(a) Given \((x, y, z)\) in terms of time.
Find \((v_x, v_y, v_z)\), \((a_x, a_y, a_z)\).

Regular calculus: MA0010, or 90, etc.

(b) Given \((a_x, a_y, a_z)\) in terms of \(t\),
and sometimes \((v_x, v_y, v_z)\), \((x, y, z)\).
Find \((v_x, v_y, v_z)\), \((x, y, z)\).

To do this, we will need to solve
"differential equations"—e.g. \(\frac{d^2 x}{dt^2} + x = 0\).

Solve by hand, or with MATLAB.
Background

Common dynamics problem:

Given: (i) Acceleration $a = f(x, v, t)$

(ii) Speed $v$ and position $x$ at time $t = 0$

Find: speed $v$ and distance traveled $x$ for $t > 0$

Approach: Solve the differential equations

$$\frac{dx}{dt} = v$$
$$\frac{dv}{dt} = v \frac{dv}{dx} = \frac{d^2x}{dt^2} = f(x, v, t)$$

Some simple cases can be solved by hand.

(For harder equations we have to use MATLAB)

Some equations can be solved using separation of variables
Separation of variables

Three cases:

1. Acceleration is a known function of time $a = f(t)$
   
   Example: Rocket in space, with thrust that decreases with time (for $0 < t < T$)
   
   $a(t) = F_0 (1 - t/T) / m$

2. Acceleration depends on speed (and time) $a = f(v)g(t)$
   
   Example: Dust particle dropping vertically with air resistance
   
   $a(v) = g - cv / m$

3. Acceleration depends on position (and speed) $a = f(x)g(v)$
   
   Example: Mass on a spring
   
   $a(x) = -kx / m$
Separation of variables: Case 1

Acceleration is a known function of time \( a = f(t) \)

\[
\frac{dv}{dt} = f(t) \quad \frac{dx}{dt} = v
\]

Initial condition: \( x(t = 0) = x_0 \quad v(t = 0) = v_0 \)

Calculating \( v \)

Step 1: ‘Separate variables’ \( \frac{dv}{f(t)} = dt \)

Step 2: Integrate both sides \( \int_{v_0}^{v} dv = \int_{0}^{t} f(t) dt \)

Example \( a = F_0 (1 - t / T) / m \) \( 0 < t < T \)

\[
\begin{align*}
\text{Step 1} & \quad \frac{dv}{dt} = \left\{\frac{F_0 (1 - t / T)}{m}\right\} dt \\
\text{Step 2} & \quad \int_{v_0}^{v} dv = \int_{0}^{t} \left\{\frac{F_0 (1 - t / T)}{m}\right\} dt \\
& \quad \Rightarrow v = v_0 + \frac{F_0}{m} \left\{t - t^2 / (2T)\right\}
\end{align*}
\]

This means substitute the limits
Separation of variables: Case 1

Acceleration is a known function of time \( a = f(t) \)

\[
\frac{dv}{dt} = f(t) \quad \frac{dx}{dt} = v
\]

Initial condition: \( x(t=0) = x_0 \quad v(t=0) = v_0 \)

Calculating \( x \)

Step 1: ‘Separate variables’

\[
dx = v(t) \, dt
\]

Step 2: Integrate both sides

\[
\int_{x_0}^{x} dx = \int_{0}^{t} v(t) \, dt
\]

Example

\( a = F_0 (1 - t / T) / m \quad v = v_0 + F_0 \left\{ t - t^2 / (2T) \right\} / m \quad 0 < t < T \)

Step 1

\[
dx = \left( v_0 + F_0 \left\{ t - t^2 / (2T) \right\} / m \right) \, dt
\]

Step 2

\[
\int_{x_0}^{x} dx = \int_{0}^{t} \left( v_0 + F_0 \left\{ t - t^2 / (2T) \right\} / m \right) \, dt
\]

\[
\Rightarrow \left[ x \right]_0^x = \left[ v_0 t + F_0 \left\{ t^2 / 2 - t^3 / (6T) \right\} / m \right]_0^t
\]

\[
\Rightarrow x = v_0 t + F_0 \left\{ t^2 / 2 - t^3 / (6T) \right\} / m
\]
Separation of variables: Case 1

Graphical Method for $a, v, x$ all functions of time

- Speed is the slope of the distance-v-time curve
- Distance is the area under the speed-v-time curve
- Acceleration is the slope of the speed-v-time curve
- Speed is the area under the acceleration-v-time curve
Separation of variables: Case 2

Acceleration depends on speed (and time) \( a = f(v)g(t) \)

\[
\frac{dv}{dt} = f(v)g(t) \quad \frac{dx}{dt} = v
\]

Initial condition: \( x(t = 0) = x_0 \quad v(t = 0) = v_0 \)

Calculating \( v \)

Step 1: ‘Separate variables’ \( \frac{dv}{f(v)} = g(t)dt \)

Step 2: Integrate both sides

\[
\int_{v_0}^{v} \frac{dv}{f(v)} = \int_{0}^{t} g(t)dt
\]

Example \( a(v) = g - cv / m \)

\[
\frac{dv}{g - cv / m} = dt
\]

\[
\int_{v_0}^{v} \frac{dv}{g - cv / m} = \int_{0}^{t} dt \quad \Rightarrow \quad \left[ -\frac{m}{c} \log \left( \frac{g - cv / m}{g - cv_0 / m} \right) \right]_{v_0}^{v} = t
\]

\[
\Rightarrow -\frac{m}{c} \log \left( \frac{g - cv / m}{g - cv_0 / m} \right) = t \quad \Rightarrow \quad v = \frac{mg}{c} - \left( \frac{mg}{c} - v_0 \right) \exp(-ct / m)
\]
Separation of variables: Case 2

Acceleration depends on speed (and time) \( a = f(v)g(t) \)

\[
\frac{dv}{dt} = f(v)g(t) \quad \frac{dx}{dt} = v \quad \text{Initial condition: } x(t = 0) = x_0 \quad v(t = 0) = v_0
\]

Calculating \( x \)

Step 1: ‘Separate variables’ \( dx = v(t)dt \)

Step 2: Integrate both sides \( \int x = \int v(t) dt \)

Example \( a(v) = g - cv / m \quad v = mg / c - (mg / c - v_0) \exp(-ct / m) \)

\[
\text{Step 1} \quad dx = (mg / c - (mg / c - v_0) \exp(-ct / m)) dt \\
\text{Step 2} \quad \int x = \int (mg / c - (mg / c - v_0) \exp(-ct / m)) dt \\
\quad \Rightarrow \quad [x]_0^x = [mgt / c + (m / c)(mg / c - v_0) \exp(-ct / m)]_0^t \\
\quad \Rightarrow \quad x = x_0 + mgt / c + (m / c)(mg / c - v_0)\{\exp(-ct / m) - 1\}
Separation of variables: Case 3

Acceleration depends on position (and speed) \( a = f(x)g(v) \)

\[
\frac{dv}{dt} = f(x)g(v) \quad \frac{dx}{dt} = v \quad \text{Initial condition: } x(t=0) = x_0 \quad v(t=0) = v_0
\]

Calculating \( v \)

Step 1: Rewrite acceleration in terms of \( x \)

\[
\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = \frac{dv}{dx} v = f(x)g(v)
\]

Step 2: ‘Separate variables’

\[
\frac{v dv}{g(v)} = f(x) dx
\]

Step 3: Integrate both sides

\[
\int_{v_0}^{v} \frac{v dv}{g(v)} = \int_{x_0}^{x} f(x) dx
\]

Example \( a = -kx \big/ m \)

Steps 1&2 \( v dv = (-kx \big/ m) dx \)

Step 3

\[
\int_{v_0}^{v} v dv = \int_{x_0}^{x} (-kx \big/ m) dx \Rightarrow \left[ \frac{v^2}{2} \right]_{v_0}^{v} = \left[ -\frac{kx^2}{2m} \right]_{x_0}^{x}
\]

\[
\Rightarrow v = \sqrt{v_0^2 + \frac{kx_0^2}{m}} - \frac{kx^2}{m}
\]
Separation of variables: Case 3

Acceleration depends on position (and speed) \( a = f(v)g(x) \)

\[
\frac{dv}{dt} = f(x)g(v) \quad \frac{dx}{dt} = v(x) \quad \text{Initial condition: } x(t=0) = x_0 \quad v(t=0) = v_0
\]

Calculating \( x \)

Step 1: ‘Separate variables’

\[
\frac{dx}{v(x)} = dt
\]

Step 2: Integrate both sides

\[
\int_{x_0}^{x} \frac{dx}{v(x)} = \int_{0}^{t} dt
\]

Example

\( a = -kx / m \) \quad \( v = \sqrt{v_0^2 + kx_0^2 / m - kx^2 / m} \)

Step 1

\[
\frac{dx}{\sqrt{v_0^2 + kx_0^2 / m - kx^2 / m}} = dt
\]

Step 2

\[
\int_{x_0}^{x} \frac{dx}{\sqrt{v_0^2 + kx_0^2 / m - kx^2 / m}} = \int_{0}^{t} dt \Rightarrow \frac{1}{\sqrt{k/m}} \left[ \sin^{-1} \left( \frac{x}{\sqrt{mv_0^2 / k + x_0^2}} \right) \right]_{x_0}^{x} = t
\]

\[\Rightarrow x = \sqrt{mv_0^2 / k + x_0^2} \sin \left( \sqrt{(k/m)} t + \sin^{-1} \left( \frac{x_0}{\sqrt{mv_0^2 / k + x_0^2}} \right) \right)\]
Final note

We use the same calculus in many other applications.

Example 1: Motion along a curved path

Differential equations
\[
\frac{dV}{dt} = V \frac{dV}{ds} = a_t(s, V, t) \quad \frac{ds}{dt} = V
\]

Replaces \( x \)

Replaces \( v \)

Example 2: Rotation about a fixed axis

Differential equations
\[
\frac{d\omega}{dt} = \omega \frac{d\omega}{d\theta} = \alpha(\theta, \omega, t) \quad \frac{d\theta}{dt} = \omega
\]

Replaces \( x \)

Replaces \( a \)

Replaces \( v \)
2.4 Example: Straight line motion with constant acceleration

A particle has constant acceleration \( \mathbf{a} = ai \)
At time \( t=0 \) it has velocity \( \mathbf{v} = v_0 i \) and position \( \mathbf{r} = x_0 i \)
Find \( \mathbf{v}(t), \mathbf{v}(x) \) and \( \mathbf{r}(t) \)

Use calculus formulas

\[
\frac{dv}{dt} = a \Rightarrow \int_{v_0}^{v} dv = \int_0^t a \, dt \Rightarrow v - v_0 = a \, t
\]

\( \Rightarrow V(t) = v_0 + a \, t \)

\[
\frac{dx}{dt} = a \Rightarrow \int_{x_0}^{x} dx = \int_{v_0}^{v} a \, dv \Rightarrow \left[ \frac{1}{2} v^2 \right]_{v_0}^{v} = a \, (x - x_0)
\]

\( \Rightarrow \frac{1}{2} V^2 - \frac{1}{2} v_0^2 = a \, (x - x_0) \)

\( \Rightarrow V(x) = \sqrt{v_0^2 + 2a \, (x - x_0)} \)
\[ \frac{dx}{dt} = v = v_0 + at \Rightarrow \int_{x_0}^{x} dx = \int_{0}^{t} (v_0 + at) \, dt \]

\[ \Rightarrow x - x_0 = v_0 t + \frac{1}{2} at^2 \]

\[ x = x_0 + v_0 t + \frac{1}{2} at^2 \]

"Constant acceleration formulas"

NB: Use these only if \( a \) is constant

Otherwise separate variables or use MATLAB
2.5 Example: Toronto high-speed walkway

Standing passenger has constant accel

Given information:
- Total length $L = 912$ ft,
- $V_1 = 125$ ft/min
- $V_2 = 400$ ft/min
- Travel time $A \rightarrow B = t_1$ approx 10 sec
- Travel time $C \rightarrow D = t_1$ approx 10 sec

Calculate:
(a) Acceleration $a$
(b) Time of travel $T$ from $A \rightarrow D$

(a) Use constant accel formulas between $A$ & $B$

$$V_x = V_i + a \cdot t_1 \quad \Rightarrow \quad a = \frac{V_x - V_i}{t_1} = \frac{400 - 125}{60 \times 10}$$

$$\Rightarrow \quad a = 0.46 \text{ ft/s}^2$$

(approx $0.015 \text{ g}$)

$g = 32 \text{ ft/s}^2$
(b) Use graphical method

Distance = area under graph

\[ L = V_2 (T - 2t_1) + 2 \left\{ \frac{(V_1 + V_2)}{2} \cdot b_1 \right\} \]

Solve for \( T \):
\[ T = \frac{L}{V_2} + 2t_1 - \frac{(V_1 + V_2)}{V_2} t_1 \]

Substitute numbers
\[ T = 142.93 \text{ s} \]

(compare to 437 s @ speed \( V_1 \))
2.5 Example: Straight line motion with variable acceleration

Aircraft starts from rest.

Acceleration \[ a = \frac{F_0}{m} \left( 1 - \frac{v}{v_0} \right) \]

Must reach speed \( v_{TO} \) to take off

1. Find a formula for speed as a function of time.
2. Find a formula for distance traveled as a function of time.
3. Find a formula for the minimum length of runway required to takeoff

Use separation of variables

\[ \frac{dv}{dt} = a = \frac{F_0}{m} \left( 1 - \frac{v}{v_0} \right) \Rightarrow \int_0^v \frac{dv}{1 - v/v_0} = \int_0^t \frac{F_0}{m} \, dt \]

\[ \Rightarrow \left[ -v_0 \log \left( 1 - \frac{v}{v_0} \right) \right]_0^v = \frac{F_0}{m} t \]

\[ \Rightarrow V = v_0 \left( 1 - \exp \left\{ -\frac{F_0}{m v_0} t \right\} \right) \]
\[
\frac{dx}{dt} = V = V_0 \left( 1 - \exp \left\{ \frac{-F_0 \ t}{mV_0} \right\} \right)
\]

\[
\Rightarrow \int_0^x dx = \int_0^t V_0 \left( 1 - \exp \left\{ \frac{-F_0 \ t}{mV_0} \right\} \right) dt
\]

\[
\Rightarrow x = \left[ V_0 t + \frac{mV_0^2}{F_0} \exp \left\{ \frac{-F_0 \ t}{mV_0} \right\} \right]_0^t
\]

\[
\Rightarrow x = V_0 t + \frac{mV_0^2}{F_0} \left( \exp \left\{ \frac{-F_0 \ t}{mV_0} \right\} - 1 \right)
\]
(3) Notes: (a) Aircraft must reach $V_{TO}$ before end of runway
(b) We can find time to reach $V_{TO}$ from part (1); then find distance traveled from (2)

$$V_{TO} = V_0 \left(1 - \exp \left\{ -\frac{F_0 t}{m V_0} \right\} \right)$$

Note: $m V_0^2 \left( \exp \left\{ -\frac{F_0 t}{m V_0} \right\} - 1 \right) = -m V_0 V_{TO} \frac{F_0}{F_0}$

Now solve for $t$

$$t = -m V_0 \log \left(1 - \frac{V_{TO}}{V_0} \right)$$

Finally

$$x = -m V_0^2 \log \left(1 - \frac{V_{TO}}{V_0} \right) - m V_0 V_{TO} \frac{F_0}{F_0}$$

Note: $V_{TO} < V_0$ and $\log \beta < 0$ for $\beta < 1$

- first term is positive!
2.7 Example: Integrating/differentiating acceleration/position with MATLAB

Accelerometers on a quadcopter and a radio positioning system measure position, velocity, and acceleration at a series of successive times. Data stored in ‘comma separated value’ (csv) file.

Use MATLAB to read the file, and
(a) integrate the acceleration \((a_x, a_y, a_z)\)
(b) Differentiate the position \((x, y, z)\)
to find \((v_x, v_y, v_z)\). Compare with measured velocity.

CSV files contain numbers separated by commas:
1, 2, 3, 4
5, 6, 7, 8
etc...
Will open in EXCEL

Data columns in example file are \(time, x, y, z, v_x, v_y, v_z, a_x, a_y, a_z\) (in SI units)

To read a csv file use:

\[
\text{data} = \text{csvread ('filename.csv')}
\]

Matrix of data (rows, cols)
Integrating acceleration data with MATLAB

Algorithm: "Trapezoidal Integration"

\[
\{ \text{Acceleration data} \} \quad \text{plotted with 'plot')}
\]

\[
\{ \text{Calculated velocity} \}
\]

MATLAB has a built-in function for this

\[
\overrightarrow{V} = \text{cumtrapz} \left[ \overrightarrow{t} \text{ values}, \overrightarrow{a \text{ values}} \right]
\]

Vector of \( V \) values

Vector of \( t \) values

Vector of \( a \) values
Differentiating positions with MATLAB

\[ V_i = \frac{X_{i+1} - X_i}{t_{i+1} - t_i} \]

\[ T_i = \frac{t_i + t_{i+1}}{2} \]

Find \( T_i, V_i \) for each \( i \) using a loop
function process_quadcopter_data
    close all
    data = csvread('demo_flight_data.csv');
    time = data(:,1); x = data(:,2); y = data(:,3); z = data(:,4);
    ax = data(:,8); vx = data(:,5);
    plot3(x,y,z);
    grid on;
    title('Trajectory'); xlabel('x (m)'); ylabel('y (m)'); zlabel('z (m)');
    figure
    plot(time,ax);
    grid on;
    title('Acceleration'); xlabel('time (s)'); ylabel('a_x (m/s^2)');
    v_integrated = cumtrapz(time,ax);
    figure
    plot(time,v_integrated,'DisplayName','Integrated a_x')
    hold on
    plot(time,vx,'DisplayName','v_x from quadcopter')
    for i = 1:length(time) - 1
        T(i) = (time(i+1) + time(i))/2;
        v_differentiated(i) = (x(i+1)-x(i))/(time(i+1)-time(i));
    end
    plot(T,v_differentiated);
    v_smoothed = smooth(v_differentiated);
    plot(T,v_smoothed,'DisplayName','Differentiated x')
    grid on;
    title('Velocity Calculations'); xlabel('time (s)'); ylabel('v_x (m/s)');
    legend(gca,'show');
end