Episode 7

Energy Relations for Conservative Systems of Particles

ENGN0040: Dynamics and Vibrations
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Topics for today's class

Energy relations for conservative systems of particles

1. Conservative, non-conservative and workless forces
2. Potential energy of a conservative force
3. Work-Power-PE-KE relation for a conservative system of particles
4. Applications
4.2 Energy relations for conservative systems of particles

4.2.1 Conservative, non-conservative and workless forces

Recall: Work done by a force

\[ W = \int_{r_0}^{r_1} \mathbf{F} \cdot d\mathbf{r} \]

Definition: For a "conservative" force, \( W \) is equal for all paths from \( r_0 \rightarrow r_1 \).

\[ W = 0 \] for a "workless" force

Conservative
- Gravity
- Electrostatic forces
- Inter-molecular forces
- Forces exerted by springs

Non-Conservative
- Friction
- Air resistance

Workless
- Lift force
- Reaction forces
- Lorenz force
4.2.2 Potential energy of a conservative force

Definition

\[ U(r) = - \int_{r_0}^{r} F \cdot dr + C \]

\( r_0 \) & C are arbitrary; choose them to make \( U \) simple

Inverse relation

\[ F = -\nabla U = - \left\{ \frac{\partial U}{\partial x} i + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial z} k \right\} \]

Note - signs!
4.2.3: Example: Calculating the potential energy of gravity (between planets)

\[ F = -\frac{GMm}{r^2} \hat{e}_r \]

\[ G = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \]

Do integral in \( \hat{e}_r, \hat{e}_\theta, \hat{e}_z \) basis

\[ U(r) = -\int_{r_0}^{r} -\frac{GMm}{r^2} \hat{e}_r \cdot dr \hat{e}_r + C \]

\[ = \int_{r_0}^{r} \frac{GMm}{r^2} dr + C = \left[ -\frac{GMm}{r} \right]_{r_0}^{r} + C \]

\[ U = -\frac{GMm}{r} + \frac{GMm}{r_0} + C \]

Choose \( C = -\frac{GMm}{r_0} \)

\[ r = \sqrt{x^2 + y^2 + z^2} \]
4.2.4: Example: The potential energy of two neighboring Cheerios floating in milk is

\[ U \approx E_0 \log \left( \frac{r}{L_0} \right) \quad r = \sqrt{x^2 + y^2} \]

where \( E_0, L_0 \) are constants

Find a formula for the force acting between them

\[ \text{Formula } \mathbf{F} = - \nabla U \]

Use chain rule

\[ \mathbf{F} = - \left\{ \frac{\partial U}{\partial r} \frac{\partial r}{\partial x} \mathbf{i} + \frac{\partial U}{\partial r} \frac{\partial r}{\partial y} \mathbf{j} \right\} \]

Note \( \frac{\partial r}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2}} \cdot 2x = \frac{x}{r} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \)

\[ \mathbf{F} = \frac{\partial U}{\partial r} \left\{ - \left( \frac{x \mathbf{i} + y \mathbf{j}}{r} \right) \right\} = \frac{\partial U}{\partial r} \left( - \frac{r}{r^2} \right) \]

magnitude \quad \text{Unit vector} \quad \text{direction}
For cheerios $\frac{\partial U}{\partial r} = \frac{E_0}{r} \Rightarrow F = \frac{E_0}{|r|} \left( \frac{-r}{|r|} \right)$

In general for a PE of form $U(r)$

1. Magnitude is $\frac{\partial U}{\partial r}$

2. Direction is towards or away from origin

\[ \frac{\partial U}{\partial r} > 0 \Rightarrow \text{force acts towards origin} \]

\[ \frac{\partial U}{\partial r} < 0 \Rightarrow \text{force acts away from origin} \]
# 4.2.5: Table of potential energies for common forces

<table>
<thead>
<tr>
<th>Type of force</th>
<th>Force vector</th>
<th>Potential energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity acting on a particle near earth's surface</td>
<td>$\mathbf{F} = -mg \mathbf{j}$</td>
<td>$U = mgv$</td>
</tr>
<tr>
<td>Gravitational force exerted on mass $m$ by mass $M$ at the origin</td>
<td>$\mathbf{F} = -\frac{GMm}{r^3} \mathbf{r}$</td>
<td>$U = -\frac{GMm}{r}$</td>
</tr>
<tr>
<td>Force exerted by a spring with stiffness $k$ and unstretched length $L_0$</td>
<td>$\mathbf{F} = -k(r - L_0) \frac{\mathbf{r}'}{r}$</td>
<td>$U = \frac{1}{2} k(r - L_0)^2$</td>
</tr>
<tr>
<td>Force acting between two charged particles</td>
<td>$\mathbf{F} = \frac{Q_1 Q_2}{4\pi \varepsilon r^2} \mathbf{r}$</td>
<td>$U = \frac{Q_1 Q_2}{4\pi \varepsilon r}$</td>
</tr>
<tr>
<td>Force exerted by one molecule of a noble gas (e.g., He, Ar, etc) on another (Lennard Jones potential). $a$ is the equilibrium spacing between molecules, and $E$ is the energy of the bond.</td>
<td>$\mathbf{F} = 12 \frac{E}{a} \left[ \left( \frac{a}{r} \right)^{13} - \left( \frac{a}{r} \right)^{7} \right] \frac{\mathbf{r}}{r}$</td>
<td>$U = E \left[ \left( \frac{a}{r} \right)^{12} - 2 \left( \frac{a}{r} \right)^{6} \right]$</td>
</tr>
</tbody>
</table>
4.2.6 Energy equation for a conservative system

- \( R_{ij} \): Force exerted on particle i by particle j
- \( F_{i}^{\text{ext}} \): External force on particle i
- \( r_i \): Position of particle i
- \( v_i \): Velocity of particle i

\( R_{ij} \): “Internal” forces  \( F_{i}^{\text{ext}} \): “external” forces

Definition: A system is conservative if all \( R_{ij} \) are conservative.

Let

\[ R_{ij} = - \frac{\partial U_{ij}}{\partial r_i} \]
Definitions

1. Total PE of system  $U_{\text{Tot}} = \sum \ U_{ij}$
   \textit{internal forces}

2. Total KE of system  $T_{\text{Tot}} = \sum \ \frac{1}{2} m_i \ V_i \ V_i^2$
   \textit{particles}

3. Power of external forces  $P_{\text{ext}} = \sum \ F_{i}^{\text{ext}} \cdot V_i$
   \textit{forces}

4. Total external work done on system
   \[ \Delta W_{\text{ext}} = \int_{t_0}^{t_1} P_{\text{ext}} \, dt \]
Energy Equations

Power - Energy relation

\[ P^{ext} = \frac{d}{dt} \left\{ U^{tot} + T^{tot} \right\} \]

Work - energy relation

\[ \Delta W^{ext} = U^{tot}_{t=t_1} + T^{tot}_{t=t_1} - (U^{tot}_{t=t_0} + T^{tot}_{t=t_0}) \]

Special Case \[ \Delta W^{ext} = 0 \]

\[ U^{tot}_{t=t_1} + T^{tot}_{t=t_1} = U^{tot}_{t=t_0} + T^{tot}_{t=t_0} \]

"Total energy is conserved"
4.2.7: Example: The pendulum is stationary in its upright configuration. Following a small disturbance it falls over. Calculate the magnitudes of the velocity and acceleration of the mass when it reaches its lowest point.

Approach:
1. System = earth + pendulum
2. Conservative, no ext forces ⇒ ∆W_{ext} = 0

Hence \( T_1 + U_1 = T_0 + U_0 \)

State (0) \( T_0 = 0 \) \( U_0 = mgL \)
State (1) \( T_1 = \frac{1}{2} m V^2 \) \( U_1 = -mgL \)

\[ \Rightarrow \frac{1}{2} m V^2 - mgL = mgL \Rightarrow V = 2\sqrt{gL} \]

Acceleration Circular motion ⇒ \( a = \frac{dV}{dt} \) \[ \frac{V^2}{R} = \frac{V^2}{L} \Rightarrow 10a = 4g \]
4.2.8: Example: A spring has the force-length relation shown.
(a) Find a formula for its potential energy
(b) A mass $m > k d_0 / g$ is suspended from the spring. Find the value of extension $x$ in the static equilibrium configuration
(c) The mass released from rest with $x=0$. Find the maximum value of $x$ and the acceleration of the mass at this instant.

\[ U = -\int_{-d_0}^{x} F \cdot \, d\ell \quad F = -k(x + d_0) \hat{i} \]

\[
\Rightarrow U = \int_{0}^{x} k(x + d_0) \, dx \Rightarrow U = \frac{1}{2} k x^2 + k d_0 x
\]

(b) Statics

\[ F_s = mg \Rightarrow k(x + d_0) = mg \]

\[
\Rightarrow x = \frac{mg - d_0}{k}
\]
(c) System = earth + spring + mass
Conservative, no ext forces
\[ T_1 + U_1 = T_0 + U_0 \]

(0) \[ T_0 = 0 \quad U_0 = -mgd_0 \]

(1) \[ T_1 = 0 \quad U_1 = -mg(d_0 + x) + \frac{1}{2}kx^2 + kdx \]

\[ \Rightarrow -mg(d_0 + x) + kx(\frac{x}{2} + d_0) = -mgd_0 \]

\[ \Rightarrow x = 2\left(\frac{mg}{k} - d_0\right) \]

Twice static deflection

Acceleration
Use \[ F = ma \]

\[ ma = F_3 - mg = k(x + d_0) - mg \]

\[ \Rightarrow a = g - \frac{kdx}{m} \]
Spring Drop Experiment

Coiled spring
h = 22.5cm
Predicted to be equal

33.5cm
Static equilibrium
h = 56cm

33cm
Max deflection
h = 89cm

Prediction
Spring stiffness: 3.5 N/m
Spring length at zero force 6.5cm
Total mass 142.8 grams

\[ a = g - kd_0 / m = 0.84g \]
4.2.8: Example: A dynamic climbing rope has a force-elongation relation that can be approximated by \( F = F_0 \varepsilon^{3/2} \)

(a) Find a formula for the potential energy of a rope with length \( L \)
(b) A climber with mass \( m \) is tethered by a rope with length \( L \). The climber falls a distance \( h \) before the rope begins to stretch, and is brought to rest as the rope stretches. Show that the fractional change in length of the rope \( \varepsilon \) is related to the fall factor \( f = \frac{h}{L} \) by

\[
f = \frac{2F_0}{5mg} \varepsilon^{5/2} - \varepsilon
\]

(c) Find the value of \( F_0 \) necessary to pass the UIAA standard

(a) Think of rope as non-linear spring

\[
F = -F_0 \varepsilon^{3/2} \quad \varepsilon = \frac{x}{L}
\]

\[
U = -\int_{F_0}^{x} F \cdot dx = \int_{0}^{x} F_0 \left( \frac{x}{L} \right)^{3/2} dx
\]

\[
\Rightarrow U = \frac{2}{5} F_0 L \left( \frac{x}{L} \right)^{5/2} = \frac{2}{5} F_0 L \varepsilon^{5/2}
\]
(b) System = earth + rope + climber

Conservative, no ext force

\[ T_1 + U_1 = T_0 + U_0 \]

(0) \[ T_0 = 0 \quad U_0 = 0 \]

(1) \[ T_1 = 0 \]

\[ U_1 = -mg(h + x) + \frac{2}{5} F_0 L \left( \frac{x}{L} \right)^{5/2} \]

\[ \Rightarrow mgh = \frac{2}{5} F_0 L \left( \frac{x}{L} \right)^{5/2} - mgx \]

\[ \Rightarrow \quad \frac{h}{L} = \frac{2}{5} \frac{F_0}{mg} \left( \frac{x}{L} \right)^{5/2} - \frac{x}{L} \]

\[ \Rightarrow \quad f = \frac{2}{5} \frac{F_0}{mg} \varepsilon^{5/2} - \varepsilon \]
(c) UIAA standard

for $m = 80 \text{ kg}$, $f = 1.75$.

(a) $\varepsilon \leq 0.4$

(b) Force in rope $< 12 \text{ kN}$

We have $f = \frac{2}{5} \frac{F_0}{mg} \varepsilon^{5/2} - \varepsilon$

Use (a) $\Rightarrow$ $F_0 = 5 \left( f + \varepsilon \right) \frac{mg}{\varepsilon^{5/2}} = 42 \text{ kN}$

Check (b) $F = F_0 \varepsilon^{3/2} = 11 \text{ kN}$

Postscript: ropes in example just pass standard.