Episode 7

Energy Relations for Conservative Systems of Particles

ENGN0040: Dynamics and Vibrations
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Topics for today's class

Energy relations for conservative systems of particles

1. Conservative, non-conservative and workless forces
2. Potential energy of a conservative force
3. Work-Power-PE-KE relation for a conservative system of particles
4. Applications
4.2 Energy relations for conservative systems of particles

4.2.1 Conservative, non-conservative and workless forces

Recall: Work done by a force

\[ W = \int_{r_0}^{r_1} F \cdot dr \]

Definition: For a “conservative” force \( W \) is equal for all paths from \( r_0 \) to \( r_1 \),

\[ W = 0 \] for a “workless” force

Conservative
- Gravity
- Electrostatic forces
- Inter-molecular forces
- Forces exerted by springs

Non-Conservative
- Friction
- Air resistance

Workless
- Lift force
- Reaction forces
- Lorenz force
4.2.2 Potential energy of a conservative force

**Definition**

\[ U(r) = - \int_{r_0}^{r_1} F \cdot dr + C \]

\( r_0 \) & \( C \) are arbitrary; choose them to make \( U \) simple

**Inverse relation**

\[ F = - \nabla U = - \left\{ \frac{\partial U}{\partial x} i + \frac{\partial U}{\partial y} j + \frac{\partial U}{\partial z} k \right\} \]

Note - signs!
4.2.3: Example: Calculating the potential energy of gravity (between planets)

\[ F = -\frac{GMm}{r^2}e_r \]

\[ G = 6.674 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \]

**Formula**

\[ U = -\int_{r_0}^{r} \mathbf{E} \cdot d\mathbf{r} + C \]

Do integral in \( e_r, e_\theta \) basis

\[ U(r) = -\int_{r_0}^{r} \frac{GMm}{r^2} e_r \cdot d\mathbf{r} e_r + C \]

\[ = \int_{r_0}^{r} \frac{GMm}{r^2} dr + C = \left[ -\frac{GMm}{r} \right]_{r_0}^{r} + C \]

\[ U = -\frac{GMm}{r} + \frac{GMm}{r_0} + C \]

Choose \( C = \frac{-GMm}{r_0} \)

\[ U = -\frac{GMm}{r} \]

\[ r = \sqrt{x^2 + y^2 + z^2} \]
4.2.3: Example: Calculating the potential energy of gravity (between planets)

\[ F = -\frac{GMm}{r^2} e_r \]

\[ G = 6.674 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} \]

**Formula**

\[ U = -\int_{r_o}^{r} \frac{F}{r} \, dr + C \]

**Do integral in \( e_r, e_\theta \) basis**

\[ U = -\int_{r_o}^{r} \frac{GMm}{r^2} e_r \cdot dr \cdot e_r + C \]

\[ = \int_{r_o}^{r} \frac{GMm}{r^2} \, dr + C = \left[ \frac{-GMm}{r} \right]_{r_o}^{r} + C \]

\[ = \frac{-GMm}{r} + \frac{GMm}{r_o} + C \]

**Choose** \( C = -\frac{GMm}{r_o} \)

\[ U = -\frac{GMm}{r} \]
4.2.4: Example: The potential energy of two neighboring Cheerios floating in milk is

\[
U \approx E_0 \log \left( \frac{r}{L_0} \right) \quad r = \sqrt{x^2 + y^2}
\]

where \( E_0, L_0 \) are constants.

Find a formula for the force acting between them.

Formula \( \mathbf{F} = -\nabla U \)

Use chain rule

\[
\mathbf{F} = - \left\{ \frac{\partial U}{\partial r} \frac{\partial r}{\partial x} \hat{i} + \frac{\partial U}{\partial r} \frac{\partial r}{\partial y} \hat{j} \right\}
\]

Note \( \frac{\partial r}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2}} \quad 2x = \frac{x}{r} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \)

\[
\mathbf{F} = \frac{\partial U}{\partial r} \left\{ - \left( \frac{x \hat{i} + y \hat{j}}{r} \right) \right\} = \frac{\partial U}{\partial r} \left( -\frac{r}{r} \right)
\]

magnitude \quad \text{unit vector} \quad \text{direction}
For cheerios $\frac{\partial U}{\partial r} = E_0 \Rightarrow F = \frac{E_0}{|r|} \left( \frac{-r}{|r|^3} \right)$

In general for a PE of form $U(r)$

1) **Magnitude** is $\frac{\partial U}{\partial r}$

2) **Direction** is towards or away from origin

$\frac{\partial U}{\partial r} > 0 \Rightarrow$ force acts towards origin

$\frac{\partial U}{\partial r} < 0 \Rightarrow$ force acts away from origin
### 4.2.5: Table of potential energies for common forces

<table>
<thead>
<tr>
<th>Type of force</th>
<th>Force vector</th>
<th>Potential energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity acting on a particle near earth's surface</td>
<td>$\mathbf{F} = -mg \mathbf{j}$</td>
<td>$U = mgv$</td>
</tr>
<tr>
<td>Gravitational force exerted on mass $m$ by mass $M$ at the origin</td>
<td>$\mathbf{F} = -\frac{GMm}{r^3} \mathbf{r}$</td>
<td>$U = -\frac{GMm}{r}$</td>
</tr>
<tr>
<td>Force exerted by a spring with stiffness $k$ and unstretched length $L_0$</td>
<td>$\mathbf{F} = -k(r-L_0) \frac{\mathbf{r}'}{r}$</td>
<td>$U = \frac{1}{2}k(r-L_0)^2$</td>
</tr>
<tr>
<td>Force acting between two charged particles</td>
<td>$\mathbf{F} = \frac{Q_1Q_2}{4\pi\varepsilon r^2} \mathbf{r}$</td>
<td>$U = \frac{Q_1Q_2}{4\pi\varepsilon r}$</td>
</tr>
<tr>
<td>Force exerted by one molecule of a noble gas (e.g., He, Ar, etc) on another (Lennard-Jones potential), $a$ is the equilibrium spacing between molecules, and $E$ is the energy of the bond.</td>
<td>$\mathbf{F} = 12 \frac{E}{a} \left[ \left( \frac{a}{r} \right)^{13} - \left( \frac{a}{r} \right)^{7} \right] \mathbf{r}$</td>
<td>$U = E \left[ \left( \frac{a}{r} \right)^{12} - 2 \left( \frac{a}{r} \right)^{6} \right]$</td>
</tr>
</tbody>
</table>
4.2.6 Energy equation for a conservative system

\[ R_{ij} \] Force exerted on particle i by particle j
\[ F^\text{ext}_i \] External force on particle i
\[ r_i \] Position of particle i
\[ v_i \] Velocity of particle i

Rij : “Internal” forces  \[ F^\text{ext}_i \] : “external” forces

Definition system is conservative \( \iff \) all \( R_{ij} \) are conservative

\[ \text{Let } \ R_{ij} = - \frac{\partial U_{ij}}{\partial r_i} \]
Definitions

1. Total PE of system \( U^{\text{Tot}} = \sum_{\text{internal forces}} U_{ij} \)

2. Total KE of system \( T^{\text{Tot}} = \sum_{\text{particles}} \frac{1}{2} m_i \dot{V}_i \dot{V}_i \)

3. Power of external forces \( P^{\text{ext}} = \sum_{\text{forces}} F_i^{\text{ext}} \cdot V_i^{\text{ext}} \)

4. Total external work done on system

\[ \Delta W^{\text{ext}} = \int_{t_0}^{t_1} P^{\text{ext}} \, dt \]
Energy Equations

Power-Energy relation
\[
p_{\text{ext}} = \frac{d}{dt} \left\{ U_{\text{TOT}} + T_{\text{TOT}} \right\}
\]

Work-Energy relation
\[
\Delta W_{\text{ext}} = U_{1,\text{TOT}} + T_{1,\text{TOT}} - (U_{0,\text{TOT}} + T_{0,\text{TOT}})
\]

Special Case \( \Delta W_{\text{ext}} = 0 \)
\[
U_{1,\text{TOT}} + T_{1,\text{TOT}} = U_{0,\text{TOT}} + T_{0,\text{TOT}}
\]

"Total energy is conserved"
4.2.7: Example: The pendulum is stationary in its upright configuration. Following a small disturbance it falls over. Calculate the magnitudes of the velocity and acceleration of the mass when it reaches its lowest point.

Approach:
1. System = earth + pendulum
2. Conservative, no ext forces ⇒ $\Delta W_{ext} = 0$
Hence $T_i + U_i = T_0 + U_0$

State (0) $T_0 = 0 \quad U_0 = mgL$
State (1) $T_i = \frac{1}{2} m V^2 \quad U_i = -mgL$

$\Rightarrow \frac{1}{2} m V^2 - mgL = mgL \Rightarrow V = 2\sqrt{gL}$

Acceleration Circular motion $\Rightarrow a = \frac{dV}{dt} \cdot \frac{L}{L} + \frac{V^2}{R} \Rightarrow 10a_1 = 4g$
4.2.8: **Example:** A spring has the force-length relation shown.

(a) Find a formula for its potential energy

(b) A mass \( m > \frac{kd_0}{g} \) is suspended from the spring. Find the value of extension \( x \) in the static equilibrium configuration

(c) The mass released from rest with \( x=0 \). Find the maximum value of \( x \) and the acceleration of the mass at this instant.

\[
(a) \quad U = -\int_{x_0}^{x} F \cdot dl \quad F = -k (x+d_0) \mathbf{i}
\]

\[
\Rightarrow U = \int_{0}^{x} k (x+d_0) \, dx \Rightarrow U = \frac{1}{2}kx^2 + kdx_0 x
\]

(b) **Statics**

\[
F_s = mg \Rightarrow k (x+d_0) = mg
\]

\[
\Rightarrow x = \frac{mg - d_0}{k}
\]
(c) System = earth + spring + mass
Conservative, no ext forces
\[ T_1 + U_1 = T_0 + U_0 \]

(0) \[ T_0 = 0 \quad U_0 = -mgd_0 \]
(1) \[ T_1 = 0 \quad U_1 = -mg(d_0 + x) + \frac{1}{2} kx^2 + kdx \]

\[ \Rightarrow \frac{-mg(d_0 + x)}{2} + kx(x + d_0) = -mgd_0 \]

\[ \Rightarrow x = 2 \left( \frac{mg}{k} - d_0 \right) \quad \text{Twice static deflection} \]

**Acceleration**

\[ \text{Use } F = ma \]

\[ ma = F_s - mg = k(x + d_0) - mg \]

\[ \Rightarrow a = g - \frac{kdx}{m} \]
Spring Drop Experiment

Coiled spring
$h = 22.5\text{cm}$

Predicted to be equal
$33.5\text{cm}$

Static equilibrium
$h = 56\text{cm}$

Max deflection
$h = 89\text{cm}$

Prediction
Spring stiffness: $3.5 \text{ N/m}$
Spring length at zero force $6.5\text{cm}$
Total mass $142.8 \text{ grams}$

$$a = g - kd_0 / m = 0.84g$$
4.2.8: Example: A dynamic climbing rope has a force-elongation relation that can be approximated by $F = F_0 \varepsilon^{3/2}$

(a) Find a formula for the potential energy of a rope with length $L$
(b) A climber with mass $m$ is tethered by a rope with length $L$. The climber falls a distance $h$ before the rope begins to stretch, and is brought to rest as the rope stretches. Show that the fractional change in length of the rope $\varepsilon$ is related to the fall factor $f=h/L$ by

$$f = \frac{2F_0}{5mg} \varepsilon^{5/2} - \varepsilon$$

(c) Find the value of $F_0$ necessary to pass the UIAA standard

(a) Think of rope as non-linear spring

$$F = -F_0 \varepsilon^{3/2} \quad \varepsilon = \frac{x}{L}$$

$$U = -\int_{F_0}^{\varepsilon} F \cdot d\varepsilon = \int_{0}^{x} F_0 \left(\frac{x}{L}\right)^{3/2} dx$$

$$\Rightarrow U = \frac{2}{5} F_0 L \left(\frac{x}{L}\right)^{5/2} = \frac{2}{5} F_0 L \varepsilon^{5/2}$$
(b) System = earth + rope + climber

Conservative, no ext force

\[ T_1 + U_1 = T_0 + U_0 \]

(0) \[ T_0 = 0 \quad U_0 = 0 \]

(1) \[ T_1 = 0 \]

\[ U_1 = -mg(h + x) + \frac{2}{5} F_0 L \left( \frac{x}{L} \right)^{5/2} \]

\[ \Rightarrow \quad mgh = \frac{2}{5} F_0 L \left( \frac{x}{L} \right)^{5/2} - mgx \]

\[ \Rightarrow \quad \frac{h}{L} = \frac{2}{5} \frac{F_0}{mg} \left( \frac{x}{L} \right)^{5/2} - \frac{x}{L} \]

\[ \Rightarrow \quad f = \frac{2}{5} \frac{F_0}{mg} \varepsilon^{5/2} - \varepsilon \]
(c) UIAA standard

For $m = 80 \text{ kg}$, $f = 1.75$:

(a) $\varepsilon \leq 0.4$

(b) Force in rope $< 12 \text{ kN}$

We have $f = \frac{2}{5} \frac{F_0}{mg} \varepsilon^{5/2} - \varepsilon$

Use (a) $\Rightarrow$

$$F_0 = \frac{5}{2} (f + \varepsilon) \frac{mg}{\varepsilon^{5/2}} = 42 \text{ kN}$$

Check (b) $F = F_0 \varepsilon^{3/2} = 11 \text{ kN}$

Postscript: ropes in example just pass standard.

![Graph](image)