Episode 13

Free Vibration of Damped Systems

ENGN0040: Dynamics and Vibrations
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Topics for today's class

Free Vibration of Damped Systems

1. Modeling energy dissipation – the dashpot
2. Damped harmonic oscillator
3. Terminology
4. Examples
5. Measuring damping experimentally
5.5 Free Vibration of damped systems

Goal: Understand influence of energy loss on free vibrations

5.5.1 Modeling energy loss: The dashpot

Exerts a force proportional to stretch rate

\[ F_d = c \frac{dL}{dt} \]

\( c \): “Dashpot coefficient” (constant - similar to spring stiffness)

Units: Ns/m
Can combine like springs

\[ L \]

\[ c_1 \]
\[ c_2 \]

Parallel
\[ C_{\text{eff}} = C_1 + C_2 \]

Series
\[ \frac{1}{C_{\text{eff}}} = \frac{1}{C_1} + \frac{1}{C_2} \]

Power to stretch a dashpot
\[ P = F \cdot \nu = -c \frac{dL}{dt} \cdot \frac{di}{dt} \]
\[ P = -c \left( \frac{dL}{dt} \right)^2 \]
Always < 0
\Rightarrow always dissipates energy
5.5.2 Free Vibration of a 1-DOF system

Canonical vibration problem: The spring-mass system is released with speed $v_0$ from position $s_0$ at time $t=0$. Find $s(t)$.

Approach: (1) EOM
(2) Find solution in tables

EOM ($F = ma$)

$$-F_s - F_d = m \frac{d^2 s}{dt^2}$$

$$F_s = k (s - s_0) \quad F_d = c \frac{ds}{dt}$$

$$-k(s - s_0) - c \frac{ds}{dt} = m \frac{d^2 s}{dt^2}$$

$$\Rightarrow \quad \frac{m}{k} \frac{d^2 s}{dt^2} + \frac{c}{k} \frac{ds}{dt} + s = s_0$$
List of standard ODEs for vibration problems

Case I \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + x = C \]

Case II \[ \frac{1}{\alpha^2} \frac{d^2x}{dt^2} - x = -C \]

Case III \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C \]

Our eq: \[ \frac{m}{k} \frac{d^2s}{dt^2} + \frac{c}{k} \frac{ds}{dt} + s = \xi_0\]

Case IV \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + KF(t) \text{ with } F(t) = F_0 \sin \omega t \]

Case V \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left( y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right) \text{ with } y(t) = Y_0 \sin \omega t \]

Case VI \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2y}{dt^2} \text{ with } y(t) = Y_0 \sin \omega t \]

Case VII \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = K \left( \frac{\lambda^2}{\omega_n^2} \frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy}{dt} + y \right) \text{ with } y(t) = Y_0 \sin \omega t \]

\[ \omega_n = \sqrt{\frac{k}{m}} \]

\[ \xi = \frac{c}{2 \sqrt{kr}} \quad \text{“Damping Coefficient”} \]
Solution to Case III (From pdf on website)

Solution to \( \frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + 2\zeta \frac{dx}{dt} + x = C \) with \( x = x_0 \), \( \frac{dx}{dt} = v_0 \), \( t = 0 \)

Let \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) (Note absolute value)

Overdamped \( \zeta > 1 \)

\[ x(t) = C + \text{exp}(-\zeta \omega_n t) \left\{ \frac{v_0 + (\zeta \omega_n + \omega_d)(x_0 - C)}{2\omega_d} \text{exp}(\omega_d t) - \frac{v_0 + (\zeta \omega_n - \omega_d)(x_0 - C)}{2\omega_d} \text{exp}(-\omega_d t) \right\} \]

Critically Damped \( \zeta = 1 \)

\[ x(t) = C + \left\{ (x_0 - C) + [v_0 + \omega_n (x_0 - C)] t \right\} \text{exp}(-\omega_n t) \]

Underdamped \( \zeta < 1 \)

\[ x(t) = C + \text{exp}(-\zeta \omega_n t) \left\{ (x_0 - C) \cos \omega_d t + \frac{v_0 + \zeta \omega_n (x_0 - C)}{\omega_d} \sin \omega_d t \right\} \]

Recall \( \omega_n = \sqrt{\frac{k}{m}} \)

\[ 5 = \frac{C}{2 \sqrt{km}} \]
Underdamped Solution ($\zeta < 1$)

\[ x(t) = A \exp(-\zeta \omega_n t) \sin(\omega_d t + \phi) \]

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

Notes:
1. Exponentially decaying vibrations
2. Frequency $\omega_d$ (slightly smaller than $\omega_n$)
3. Decays as $\exp(-\zeta \omega_n t)$

Vibrations decay faster as $\zeta \to 1$

Fastest decay for $\zeta = 1$ ~ Critical Damping
Overdamped Solution \( s > 1 \)

\[
x(t) = \exp(-swt) \left\{ A \exp(sw't) + B \exp(-sw't) \right\}
\]

\[
w_d = wn \sqrt{s^2 - 1}
\]

\[
\Rightarrow x(t) = A \exp \left\{ -wt \left( s - \sqrt{s^2 - 1} \right) \right\} + B \exp \left\{ -wt \left( s + \sqrt{s^2 - 1} \right) \right\}
\]

Decays slowly

Decays quickly

Notes:

1. No vibrations
2. Slowest term decays faster as \( s \to 1 \)
   slower as \( s \to \infty \)

Fastest decay for \( s = 1 \)

![Graph showing fast and slow decay](image)
Engineering Implications

(1) To stop vibrations \( s > 1 \Rightarrow c > 2\sqrt{km} \)

(2) For fastest return to equilibrium

(a) Choose critical damping \( s = 1 \)

Then \( x = A \exp(-\omega_n t) \)

Hence maximize \( \omega_n \)

\[ \Rightarrow \text{maximize } \omega_n = \sqrt{\frac{k}{m}} \]

Choose \( c = 2\sqrt{km} \)
**Underdamped System**

Amplitude decays as $\exp(-5\omega_n t)$

$\Rightarrow$ increase $5\omega_n = c/(2m)$

$\Rightarrow$ increase $c$ or reduce $m$

**Overdamped System**

$x$ decays as $\exp \left\{ -\omega_n (5 - \sqrt{5^2 - 1}) t \right\}$

$\Rightarrow$ increase $\omega_n/(25) = k/(2c)$

$\Rightarrow$ increase $k$ or reduce $c$
Solving the case III vibration equation

Background: Complex Variables

Define \( i = \sqrt{-1} \) General complex number \( z = a + ib \)

Complex conjugate \( \bar{z} = a - ib \) \( \Rightarrow \) \( a = (z + \bar{z}) / 2 \) \( b = -i(z - \bar{z}) / 2 \)

Euler’s formula \( e^{i\theta} = \cos \theta + i \sin \theta \)

Polar / rectangular conversion
\[
\begin{align*}
    a + ib &= \rho e^{i\theta} \\
    \rho &= \sqrt{a^2 + b^2} \\    \theta &= \tan^{-1}(b / a) \\
    a &= \rho \cos \theta \\
    b &= \rho \sin \theta
\end{align*}
\]

Trig functions
\[
\begin{align*}
    \cos \theta &= \frac{\left(e^{i\theta} + e^{-i\theta}\right)}{2} \\
    \sin \theta &= -i\frac{\left(e^{i\theta} - e^{-i\theta}\right)}{2}
\end{align*}
\]

Calculus
\[
\begin{align*}
    \frac{d}{dt} e^{i\omega t} &= i\omega e^{i\omega t} \\
    \frac{d^2}{dt^2} e^{i\omega t} &= -\omega^2 e^{i\omega t}
\end{align*}
\]
Solving the case III vibration equation

Solve: \[ \frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C \quad x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0 \]

Guess \( x = Ae^{\lambda t} + C \) \[ \left( \frac{\lambda^2}{\omega_n^2} + \frac{2\zeta\lambda}{\omega_n} + 1 \right) Ae^{\lambda t} = 0 \] (characteristic equation)

Roots \( \lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \)

\( \zeta > 1 \) (Overdamped) \( \Rightarrow \) two real roots \( \lambda = -\zeta\omega_n \pm \omega_d \)

\( \zeta = 1 \) (Critically damped) \( \Rightarrow \) one real root \( \lambda = -\omega_n \)

\( \zeta < 1 \) (Underdamped) \( \Rightarrow \) two complex roots \( \lambda = -\zeta\omega_n \pm i\omega_d \)

General Solution:

\( \zeta > 1 \Rightarrow x = C + A_1 e^{(-\zeta\omega_n + \omega_d)t} + A_2 e^{(-\zeta\omega_n - \omega_d)t} \)

\( \zeta = 1 \Rightarrow x = C + A_1 e^{-\zeta\omega_n t} + A_2 te^{-\zeta\omega_n t} \)

\( \zeta < 1 \Rightarrow x = C + A_1 e^{(-\zeta\omega_n + i\omega_d)t} + A_2 e^{(-\zeta\omega_n - i\omega_d)t} \)

\( A_1, A_2 \) Determined by initial conditions

Note absolute value
Solving the case III vibration equation

Overdamped Solution: \( \zeta > 1 \Rightarrow x = C + A_1 e^{(-\zeta \omega_n + \omega_d)t} + A_2 e^{(-\zeta \omega_n - \omega_d)t} \)

Initial Conditions: \( x(0) = C + A_1 + A_2 = x_0 \)

\( \frac{dx}{dt} \bigg|_{t=0} = A_1 (\omega_d - \zeta \omega_n) - A_1 (\omega_d + \zeta \omega_n) = v_0 \)

\( \Rightarrow A_1 = \frac{v_0 + (\zeta \omega_n + \omega_d)(x_0 - C)}{2\omega_d} \quad A_2 = -\frac{v_0 + (\zeta \omega_n - \omega_d)(x_0 - C)}{2\omega_d} \)

\( x(t) = C + \exp(-\zeta \omega_n t) \left\{ \frac{v_0 + (\zeta \omega_n + \omega_d)(x_0 - C)}{2\omega_d} \exp(\omega_d t) - \frac{v_0 + (\zeta \omega_n - \omega_d)(x_0 - C)}{2\omega_d} \exp(-\omega_d t) \right\} \)
Solving the case III vibration equation

Critically Damped Solution: \( \zeta = 1 \Rightarrow x = C + A_1 e^{-\zeta \omega_n t} + A_2 t e^{-\zeta \omega_n t} \)

Initial Conditions:
\[
x(0) = C + A_1 = x_0
\]
\[
\left. \frac{dx}{dt} \right|_{t=0} = -\omega_n A_1 + A_2 = v_0
\]
\[
A_1 = x_0 - C \quad A_2 = v_0 + \omega_n (x_0 - C)
\]

\[
x(t) = C + \left\{ (x_0 - C) + [v_0 + \omega_n (x_0 - C)] t \right\} \exp(-\omega_n t)
\]
Solving the case III vibration equation

Underdamped Solution: \[ \zeta < 1 \Rightarrow x = C + A_1e^{(-\zeta \omega_n + i\omega_d)t} + A_2e^{(-\zeta \omega_n - i\omega_d)t} \]

Initial Conditions: \[ x(0) = C + A_1 + A_2 = x_0 \]
\[ \frac{dx}{dt} \bigg|_{t=0} = A_1(i\omega_d - \zeta \omega_n) - A_1(i\omega_d + \zeta \omega_n) = v_0 \]
\[ \Rightarrow A_1 = -i\frac{v_0 + (\zeta \omega_n + i\omega_d)(x_0 - C)}{2\omega_d} \quad A_2 = i\frac{v_0 + (\zeta \omega_n - i\omega_d)(x_0 - C)}{2\omega_d} \]

\[ x(t) = C + \exp(-\zeta \omega_n t) \left\{ (x_0 - C) \frac{1}{2} \left( e^{i\omega_d t} + e^{-i\omega_d t} \right) - \frac{v_0 + \zeta \omega_n (x_0 - C)}{\omega_d} \frac{i}{2} \left( e^{i\omega_d t} - e^{-i\omega_d t} \right) \right\} \]

Euler’s formula:

\[ x(t) = C + \exp(-\zeta \omega_n t) \left\{ (x_0 - C) \cos \omega_d t + \frac{v_0 + \zeta \omega_n (x_0 - C)}{\omega_d} \sin \omega_d t \right\} \]
5.5.3 Summary of constants used in damped vibration formulas

1. The constant \( c \) (called “dashpot coefficient” or “damping coefficient”)

2. \( \omega_n \): “Un-damped natural frequency” \( \omega_n = \sqrt{k/m} \)

3. \( \xi \): called “damping coefficient” or “damping ratio” \( \xi = c/(2\sqrt{km}) \)
   Some texts call the product \( \xi \omega_n \) the “damping ratio”

4. \( \omega_d \): “Damped natural frequency” \( \omega_d = \omega_n \sqrt{1-\xi^2} \)

Terminology can be confusing!
5.5.4: **Example:** The spring-mass system shown in the figure is critically damped.

Find the new damping factor if one spring is removed.

Formula: \[ \zeta = \frac{c}{2 \sqrt{km}} \]

"Critical damping" \[\Rightarrow \zeta = 1 \]

For original system: \[ \zeta_1 = \frac{c}{2 \sqrt{2km}} = 1 \]

For system with 1 spring removed

\[ \zeta_2 = \frac{c}{2 \sqrt{km}} = \sqrt{2} \zeta_1 = \sqrt{2} \]
5.5.5 Example: Design a thrust-stand

Constraints:
1. Mass of stand+engine 1000kg
2. Max expected thrust: 100 kN
3. Max deflection 1cm
4. Must reach equilibrium deflection as quickly as possible

Find values for $k$ and $c$.

**Approach:**
1. Statics to find $k$
2. Critical damping to find $c$

1. **Static**

$\sum F = 0 \Rightarrow F_T - F_S - F_d = 0$

$F_S = kx$

$F_d = c \frac{dx}{dt}$

Hence $F_T = kx \Rightarrow k = \frac{F_T}{x} = 10^7 N/m$
5.5.5 Example: Design a thrust-stand

Constraints:
1. Mass of stand+engine 1000kg
2. Max expected thrust: 100 kN
3. Max deflection 1cm
4. Must reach equilibrium deflection as quickly as possible

Find values for $k$ and $c$.

Critical damping $\Rightarrow S = \frac{c}{2\sqrt{km}} = 1$

$\Rightarrow c = 2\sqrt{km} = 200 \text{ kN}s/m$
5.5.6: Example: Impact of a baseball is idealized as a damped spring-mass system.

At time $t=0$, $x=0$, $\frac{dx}{dt} = v_0$

Find a formula for $x(t)$ in terms of $\zeta$, $\omega_n$.

If $\zeta$ is small, rebound occurs at approximately $x = 0$

Show that the restitution coefficient is $e \approx \exp(-\pi\zeta)$

The impact movie shows contact last for 4 millisec and $e=0.6$. The baseball has mass 0.145kg. Find $k, c$ for the baseball.

**Approach**

1. **Derive EOM** ($F=ma$)

2. **Solve**

3. **Use solution to predict rebound velocity** $\Rightarrow$ find $e$

\[ F = ma \]

\[ m \frac{d^2x}{dt^2} = F_s + F_d \]

\[ F_s = -kx \]

\[ F_d = -c \frac{dx}{dt} \]

\[ \Rightarrow m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = 0 \]
Hence \( \frac{1}{\omega_n^2} \frac{m}{k} \frac{d^2x}{dt^2} + \frac{C}{k} \frac{dx}{dt} + x = 0 \)

Case III EOM with \( \omega_n = \sqrt{\frac{k}{m}} \)

\[ 5 = \frac{C}{2\sqrt{km}} \]

We know ball rebounds \( \Rightarrow \) underdamped

\[ \Rightarrow \quad 5 < 1 \]

\[ x(t) = C + \exp(-\zeta \omega_n t) \left\{ (x_0 - C) \cos \omega_d t + \frac{v_0 + \zeta \omega_n (x_0 - C)}{\omega_d} \sin \omega_d t \right\} \]

\[ C = 0 \quad \text{Given} \quad x = 0 \quad @ \quad t = 0 \quad \Rightarrow \quad x_0 = 0 \]

\[ \frac{dx}{dt} = v_0 \quad @ \quad t = 0 \]

\[ \Rightarrow \quad x(t) = \frac{v_0}{\omega_d} \exp\left(-\zeta \omega_n t\right) \sin \omega_d t \]

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]
Understanding Solution

Restitution coefficient

\[ e = -\frac{V_i}{V_0} \]

At rebound \( x \approx 0 \)

\[ \Rightarrow \sin (W_d t_{contact}) = 0 \]

\[ \Rightarrow W_d t_{contact} = \pi \]

\[ \Rightarrow t_{contact} = \frac{\pi}{W_d} \]

\[ V = \frac{dx}{dt} = \frac{d}{dt} \left\{ \frac{V_0}{W_d} \exp(-5W_n t) \sin W_d t \right\} \]

\[ = \frac{V_0}{W_d} \exp(-5W_n t) \left\{ -5W_n \sin W_d t + W_d \cos W_d t \right\} \]
Rebound Velocity: substitute \( t = \frac{\pi}{\omega_d} \)

\[
\Rightarrow \quad V_i = -\frac{V_0}{\omega_d} \exp \left( -\frac{3\omega_n \pi}{\omega_d} \right) \omega_d
\]

Hence

\[
e = \frac{-V_i}{V_0} = \exp \left( -\frac{3\omega_n \pi}{\omega_d} \right)
\]

(Note for \( \delta << 1 \), \( \sqrt{1-\delta^2} \approx 1 \))

Hence

\[
e = \exp \left( -\pi \delta \right)
\]
Rebound velocity: substitute $t = \frac{\pi}{\omega d}$

$$\Rightarrow v_i = -\frac{v_0}{\omega d} \exp \left( -\frac{3\omega_n \pi}{\omega d} \right) \omega d$$

Hence $e = -\frac{v_i}{v_0} = \exp \left( -\frac{3\omega_n \pi}{\omega d} \right)$

$$\Rightarrow e = \exp \left( -\frac{3\pi}{\sqrt{1-\delta^2}} \right)$$

For small $\delta$, $\sqrt{1-\delta^2} \approx 1$

$$\Rightarrow e = \exp \left( -3\pi \right)$$
Finally find \( k, c \) from given data

(1) \( e = 0.6 \Rightarrow \exp(-\pi s) = 0.6 \Rightarrow s = -\frac{1}{\pi} \log(0.6) \)

\[ s = 0.16 \]

(2) \( t_{\text{contact}} = 4 \times 10^{-3} \text{ s} \quad t_{\text{contact}} = \frac{\pi}{wd} \)

\[ w_n = \frac{\pi}{t_{\text{contact}} \sqrt{1-s^2}} = 7900 \text{ rad/s} \]

\[ w_n = \sqrt{\frac{k}{m}} \Rightarrow k = mw_n^2 \quad m = 0.145 \text{ kg} \]

\[ k = 1.14 \text{ kN/m} \]

\[ s = \frac{c}{2\sqrt{km}} \Rightarrow c = 2\sqrt{km}^1 s = 4.2 \text{ Ns/m} \]
5.5.7 Measuring $W_n$ and $S$ from an impulse test

Impulse test produces a damped vibration response

Procedure

1. Find period $T$
2. Find "log decrement"

$$S = \frac{1}{n} \log \left( \frac{x(t_0)}{x(t_n)} \right)$$

Then

$$W_n = \sqrt{4\pi^2 + S^2}$$

$$S = \frac{S}{\sqrt{4\pi^2 + S^2}}$$
Proof

Damped vibration sol \( x(t) = A \exp(-\beta \omega_n t) \sin(\omega_d t + \phi) \)

Hence \( \frac{x(t_0)}{x(t_n)} = \frac{A \exp(-\beta \omega_n t_0) \sin(\omega_d t_0 + \phi)}{A \exp(-\beta \omega_n t_n) \sin(\omega_d (t_0 + nT) + \phi)} \)

\[ \Rightarrow \quad \frac{x(t_0)}{x(t_n)} = \exp(\beta \omega_n nT) \]

\[ \Rightarrow \quad \delta = \frac{1}{n} \log \left( \frac{x(t_0)}{x(t_n)} \right) = \beta \omega_n T \]

Recall \( T = \frac{2\pi}{\omega_d} \) \( \omega_d = \omega_n \sqrt{1 - \delta^2} \)

\[ \Rightarrow \quad \delta = \frac{2\pi \delta}{\sqrt{1 - \delta^2}} \quad \Rightarrow \quad \delta = \delta \quad \omega_n = \frac{\delta}{3T} = \frac{\sqrt{4\pi^2 + \delta^2}}{T} \]
Example: Fig shows vibration of a vibration isolation table.

Find $\omega_n, \xi$

Approach: Use formulas

From graph:
1. $T = 0.25 \text{ s}$
2. Log decrement $\delta = \frac{1}{2} \log \left( \frac{2.1}{0.6} \right) = 0.63$

$$\xi = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} = 0.1$$

$$\omega_n = \frac{\sqrt{4\pi^2 + \delta^2}}{T} \approx 25 \text{ rad/s}$$