Episode 15
Forced Vibrations Part 2
Base and Rotor Excitation

ENGN0040: Dynamics and Vibrations
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Topics for today's class

Forced Vibrations

1. Base excited harmonic oscillator
2. Examples
3. Rotor excited harmonic oscillator
4. Examples
5. Anti-resonant vibration isolation system
5.6.7 Base excited harmonic oscillator

Canonical base excited vibration problem: The base of the spring mass system moves harmonically \( y(t) = Y_0 \sin \omega t \).
Find steady state solution for \( s(t) \).

Approach: (1) EOM; (2) Solve (tables)

Equation of motion

\[
F - mg = -F_s - F_d = m \frac{d^2 s}{dt^2}
\]

\[
F_s = k (s - y - L_0)
\]

\[
F_d = c \frac{ds}{dt} (s - y)
\]

Hence

\[
\frac{m}{k} \frac{d^2 s}{dt^2} + \frac{c}{k} \frac{ds}{dt} + s = L_0 + y + \frac{c}{k} \frac{dy}{dt}
\]
List of standard ODEs for vibration problems

Case I \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + x = C \]

Our eq: \[ \frac{m}{k} \frac{d^2x}{dt^2} + \frac{25}{k} \frac{dx}{dt} + x = C + \frac{C}{k} (y + 25 \frac{dy}{dt}) \]

Case II \[ \frac{1}{\alpha^2} \frac{d^2x}{dt^2} - x = -C \]

Case III \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C \]

Case IV \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K F(t) \text{ with } F(t) = F_0 \sin \omega t \]

Case V \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left( y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right) \text{ with } y(t) = Y_0 \sin \omega t \]

Case VI \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2y}{dt^2} \text{ with } y(t) = Y_0 \sin \omega t \]

Case VII \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = K \left( \frac{\lambda^2}{\omega_n^2} \frac{d^2y}{dt^2} + 2\zeta \frac{dy}{dt} + y \right) \text{ with } y(t) = Y_0 \sin \omega t \]

\[ \omega_n = \sqrt{\frac{k}{m}} \quad \zeta = \frac{C}{2\sqrt{km}} \quad K = 1 \]
Solution to Case V (From pdf on website)

\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left( y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right)
\]

Initial Conditions \[ x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0 \]

Full Solution \[ x(t) = C + x_h(t) + x_p(t) \]

**Focus on this**

Steady state part (particular integral) \[ x(t) = X_0 \sin(\omega t + \phi) \]

\[
X_0 = \frac{K Y_0 \left\{ 1 + \left( 2\zeta \omega / \omega_n \right)^2 \right\}^{1/2}}{\left\{ \left( 1 - \omega^2 / \omega_n^2 \right)^2 + \left( 2\zeta \omega / \omega_n \right)^2 \right\}^{1/2}}
\]

\[
\phi = \tan^{-1} \left( \frac{-2\zeta \omega^3 / \omega_n^3}{1 - (1 - 4\zeta^2) \omega^2 / \omega_n^2} \right)
\]

Transient part (complementary integral)

Overdamped \( \zeta > 1 \)

\[ x_h(t) = \exp(-\zeta \omega_n t) \left\{ \frac{v_0 + (\zeta \omega_n + \omega_d) x_0^h}{2\omega_d} \exp(\omega_d t) - \frac{v_0^h + (\zeta \omega_n - \omega_d) x_0^h}{2\omega_d} \exp(-\omega_d t) \right\} \]

Critically Damped \( \zeta = 1 \)

\[ x_h(t) = \left\{ x_0^h + \left[ v_0^h + \omega_d x_0^h \right] t \right\} \exp(-\omega_n t) \]

Underdamped \( \zeta < 1 \)

\[ x_h(t) = \exp(-\zeta \omega_n t) \left\{ x_0^h \cos \omega_d t + \frac{v_0^h + \zeta \omega_n x_0^h}{\omega_d} \sin \omega_d t \right\} \]

\[
\omega_d = \omega_n \sqrt{1 - \zeta^2}
\]

\[ x_0^h = x_0 - C - x_P(0) = x_0 - C - X_0 \sin \phi \quad v_0^h = v_0 - \left. \frac{dx_p}{dt} \right|_{t=0} = v_0 - X_0 \omega \cos \phi \]
Steady state solution for base excited system

Steady state solution to
\[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left( y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right) \]

\[ \omega_n = \sqrt{\frac{k}{m}} \quad \zeta = \frac{c}{2\sqrt{km}} \quad K = 1 \]

\[ x(t) = X_0 \sin(\omega t + \phi) \]

\[ y(t) = Y_0 \sin \omega t \]

\[ X_0 = K \Gamma_0 M(\omega, \omega_n, \zeta) \]

\[ M = \frac{\left\{ 1 + \left( \frac{2\zeta}{\omega_n} \right)^2 \right\}^{1/2}}{\left\{ \left( 1 - \omega^2 / \omega_n^2 \right)^2 + (2\zeta \omega / \omega_n)^2 \right\}^{1/2}} \]

\[ \phi = \tan^{-1} \left( \frac{-2\zeta \omega^3 / \omega_n^3}{1 - (1 - 4\zeta^2) \omega^2 / \omega_n^2} \right) \]

Graph showing magnification M versus frequency ratio \( \omega / \omega_n \) for different values of \( \zeta \): 0.01, 0.05, 0.1, 0.15, 0.2, 0.25.
Understanding Solution

\[ x_p(t) = X_0 \sin(\omega t + \phi) \]

\[ X_0 = k \times \frac{M}{M(\omega/\omega_n, 5)} \]

\[ M = \frac{\sqrt{1 + (2.5\omega/\omega_n)^2}}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2.5\omega/\omega_n)^2}} \]

1. \( \omega < \omega_n \Rightarrow M \approx 1 \Rightarrow X_0 = Y_0 \)

2. \( \omega \approx \omega_n \Rightarrow M \approx \frac{1}{2.5} \Rightarrow X_0 = Y_0 / 2.5 \) (Resonance)

3. \( \omega > \omega_n \) \( M \approx \frac{2.5\omega_n/\omega}{1} \Rightarrow X_0 = Y_0 \cdot \frac{2.5\omega_n}{\omega} \)

Vibration isolation

For isolation: (1) \( \omega/\omega_n > \sqrt{2} \)

(2) Smaller \( S \) is better
Steady-state solution to Case V equation

Solve: \[ \frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2 \zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left( \frac{y}{\omega_n} \frac{dy}{dt} \right) \quad y = Y_0 \sin \omega t \]

Let \( y = Y_0 \text{Im}(e^{i\omega t}) \)  
Recall \( \text{Im}(z) = -i(z - \overline{z}) / 2 \)

Guess \( x(t) = C + x_p(t) \)
\[ x_p = MK \sin(\omega t + \phi) = MK \text{Im} \left\{ e^{i(\omega t + \phi)} \right\} \]

Substitute into ODE
\[ \left( 1 - \frac{\omega^2}{\omega_n^2} + i \frac{2 \zeta \omega}{\omega_n} \right) Me^{i(\omega t + \phi)} = \left( 1 + i \frac{2 \zeta \omega}{\omega_n} \right) e^{i\omega t} \]

Hence \( Me^{i\phi} = \frac{\left( 1 + i \frac{2 \zeta \omega}{\omega_n} \right)}{\left( 1 - \frac{\omega^2}{\omega_n^2} + i \frac{2 \zeta \omega}{\omega_n} \right)} = \frac{\sqrt{1 + \left( \frac{2 \zeta \omega}{\omega_n} \right)^2} e^{\tan^{-1} \left( \frac{2 \zeta \omega}{\omega_n} \right)}}{\sqrt{\left( 1 - \omega^2 / \omega_n^2 \right)^2 + \left( 2 \zeta \omega / \omega_n \right)^2} e^{\tan^{-1} \left( \frac{2 \zeta \omega / \omega_n}{1 - \omega^2 / \omega_n^2} \right)}} \]

Euler: \( a + ib = \rho e^{i\theta} \quad \rho = \sqrt{a^2 + b^2} \quad \theta = \tan^{-1} (b / a) \)

Hence \( M = \frac{\sqrt{1 + \left( \frac{2 \zeta \omega}{\omega_n} \right)^2}}{\sqrt{\left( 1 - \omega^2 / \omega_n^2 \right)^2 + \left( 2 \zeta \omega / \omega_n \right)^2}} \) 
\( \phi = \tan^{-1} \left( \frac{2 \zeta \omega}{\omega_n} \right) - \tan^{-1} \left( \frac{2 \zeta \omega / \omega_n}{1 - \omega^2 / \omega_n^2} \right) \)
5.6.8 Example: The vibration isolation system shown in the figure has 
$m=20\text{kg}, k=19.8\text{kN/m}, c=1.259\text{kN/m}$
The base vibrates harmonically with amplitude 1mm and frequency of 
100Hz. What is the steady-state amplitude of vibration of the platform?

Use formulas for $X_o$.

\[ \omega_n = \sqrt{\frac{k}{m}} = 31.46 \text{ rad/s} \]
\[ \xi = \frac{c}{2\sqrt{k/m}} = 1 \]

Magnification \[ M = \frac{\sqrt{1+(2\xi\omega/\omega_n)^2}}{\sqrt{(1-\omega^2/\omega_n)^2 + (2\xi\omega/\omega_n)^2}} = 0.1 \]

Vibration amplitude \[ X_o = kM\, Y_0 \]
$k=1$ for base excited system

\[ \Rightarrow \quad X_o = 0.1 \text{ mm} \]
5.6.9 Example: A car suspension has natural frequency $f_n = 2\, \text{Hz}$ and damping factor $\zeta = 0.2$. It drives over a road with a sinusoidal profile, with wavelength 10m and amplitude 20cm.

(a) At what car speed does max vibration amplitude occur?

(b) What is the max vibration amplitude?

(c) Redesign the suspension. Constraints:
   • Vibration amplitude must be less than 35cm at all speeds
   • At 55mph, vibration amplitude must be less than 10cm
   • Car weight 3000lb
   • Select values for $k$ and $c$

\[
\text{Approach:} \quad (1) \quad \text{Use road profile to find } y(t) \\
(2) \quad \text{Max amplitude occurs if } \omega = \omega_n
\]

Road profile: $y = Y_0 \sin \frac{2\pi z}{\lambda} \quad z = Vt$

$\Rightarrow \quad y = Y_0 \sin \left( \frac{2\pi V}{L} t \right) \quad \omega = \frac{2\pi V}{L}$
Max vibrations at $\omega = \omega_n$

$\Rightarrow \frac{2\pi V}{l} = f_n \cdot 2\pi \Rightarrow V = lf_n = 20 \text{ m/s}$

Vibration amplitude $X_0 = K\text{ m}$

At $\omega = \omega_n$, $M \approx \frac{1}{25}$, $K = 1$

$\Rightarrow X_0 = \frac{20}{(2 \times 0.2)} = 50 \text{ cm}$

To redesign the suspension

1. Note $X_0$ is determined by $M$
2. Use constraints to find form needed for $M$
3. Find $\omega_n$, $5$ to give desired $M$
4. Use formulas for $\omega_n$, $5$ to find $K$, $c$
Constraints:
- Vibration amplitude must be less than 35cm at all speeds
- At 55mph, vibration amplitude must be less than 10cm

Formula: $X_o = K \cdot M \cdot Y_o$
(K = 1)

Constraint (1): $X_o < 35 \text{cm} \quad Y_o = 20 \text{cm}$

$\Rightarrow M = \frac{X_o}{Y_o} < \frac{35}{20} = 1.75$

From graph $5 > 0.38$ for $M < 1.75$

For best isolation we want $S$ as small as possible

$\Rightarrow$ choose $S = 0.38$
Constraint (2) \( x_0 < 10 \text{ cm for } v = 55 \text{ mph} \) (25 m/s)

Recall \( \omega = \frac{2\pi v}{L} = \frac{2\pi \cdot 25}{10} = 5\pi \)

\( x_0 < 10 \Rightarrow \frac{x_0}{\gamma_0} < \frac{10}{20} \Rightarrow M < \frac{1}{2} \)

We need \( \frac{\omega}{\omega_n} > 2.1 \) for \( M < \frac{1}{2} \)

Hence \( \omega_n < \frac{\omega}{2.1} = \frac{5\pi}{2.1} \)

Choose \( \omega_n = \frac{5\pi}{2.1} \) (gives stiffest allowable suspension)
Finally recall $W_n = \sqrt{\frac{k}{m}}$

$\Rightarrow k = m W_n^2 = 168 \times 10^3 \text{ lb/ft}$

also $z = C / (2 \sqrt{km})$

$\Rightarrow C = 2 \sqrt{km} \Rightarrow z = 17 \times 10^3 \text{ lb s/ft}$
**Canonical rotor excited vibration problem:** A rotor with mass $m_0$ and length $Y_0$ rotates at steady angular velocity $\omega$. It is attached to mass $m$ which is supported by a spring and damper.

Find steady state solution for $s(t)$.

\[ F_x = \max \text{ for mass } m \]
\[ m \frac{d^2 s}{dt^2} = -F_s - F_d + H \]

\[ F_x = \max \text{ for mass } m_0 \]
\[ m_0 \frac{d^2 (s+y)}{dt^2} = -H \]

Add eqs: \( (m+m_0) \frac{d^2 s}{dt^2} + m_0 \frac{d^2 y}{dt^2} = -F_s - F_d \)

\[ F_s = k (s-L_0) \]
\[ F_d = c \frac{ds}{dt} \]

\[ \Rightarrow \frac{m+m_0}{k} \frac{d^2 s}{dt^2} + \frac{c}{k} \frac{ds}{dt} + s = L_0 - \frac{m_0}{k} \frac{d^2 y}{dt^2} \]
List of standard ODEs for vibration problems

**Case I**
\[
\omega_n^2 \frac{d^2 x}{dt^2} + x = C
\]
\[
\frac{m+m_o}{k} \frac{d^2 s}{dt^2} + \frac{C}{k} \frac{ds}{dt} + S = K0 - \frac{m_o}{k} \frac{d^2 y}{dt^2}
\]

**Case II**
\[
\alpha^2 \frac{d^2 x}{dt^2} - x = -C
\]
\[
\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2 y}{dt^2}
\]

**Case III**
\[
\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C
\]

**Case IV**
\[
\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + KF(t) \text{ with } F(t) = F_0 \sin \omega t
\]
\[
\frac{K}{\omega_n^2} = \frac{m_0}{k}
\]
\[\Rightarrow K = \frac{m_0}{k} \frac{k}{m + m_0}\]

**Case V**
\[
\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left( y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right) \text{ with } y(t) = Y_0 \sin \omega t
\]

**Case VI**
\[
\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2 y}{dt^2} \text{ with } y(t) = Y_0 \sin \omega t
\]

**Case VII**
\[
\frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = K \left( \frac{\lambda^2}{\omega_n^2} \frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy}{dt} + y \right) \text{ with } y(t) = Y_0 \sin \omega t
\]

\[
\omega_n = \sqrt{\frac{k}{m + m_o}}
\]
\[
\zeta = \frac{C}{2 \sqrt{K(m + m_o)}}
\]
\[
K = \frac{m_0}{m + m_o}
\]
Solution to Case VI (From pdf on website)

Equation \[ \frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2 y}{dt^2} \]

Initial Conditions \[ x = x_0 \quad \frac{dx}{dt} = v_0 \quad t = 0 \]

Full Solution \[ x(t) = C + x_h(t) + x_p(t) \]

Focus on this

Steady state part (particular integral) \[ x_p(t) = X_0 \sin(\omega t + \phi) \]

\[ X_0 = KY_0 M(\omega, \omega_n, \zeta) \]

\[ M = \frac{\omega^2 / \omega_n^2}{\left\{1 - \frac{\omega^2}{\omega_n^2} + \left(2\zeta \omega / \omega_n\right)^2\right\}^{1/2}} \]

\[ \phi = \tan^{-1} \frac{-2\zeta \omega / \omega_n}{1 - \omega^2 / \omega_n^2} \]

Transient part (complementary integral)

Overdamped \[ \zeta > 1 \]

\[ x_h(t) = \exp(-\zeta \omega_n t) \left\{ \frac{\nu_0^h + (\zeta \omega_n + \omega_d) x_0^h}{2\omega_d} \exp(\omega_d t) - \frac{\nu_0^h + (\zeta \omega_n - \omega_d) x_0^h}{2\omega_d} \exp(-\omega_d t) \right\} \]

Critically Damped \[ \zeta = 1 \]

\[ x_h(t) = \left\{ x_0^h + \left[ \frac{\nu_0^h}{\omega_n} \right] t \right\} \exp(-\omega_n t) \]

Underdamped \[ \zeta < 1 \]

\[ x_h(t) = \exp(-\zeta \omega_n t) \left\{ x_0^h \cos \omega_d t + \frac{\nu_0^h + \zeta \omega_n x_0^h}{\omega_d} \sin \omega_d t \right\} \]

\[ \omega_d = \omega_n \sqrt{1 - \zeta^2} \]

\[ x_0^h = x_0 - C - x_p(0) = x_0 - C - X_0 \sin \phi \quad \nu_0^h = v_0 - \left. \frac{dx_p}{dt} \right|_{t=0} = v_0 - X_0 \omega \cos \phi \]
Steady state solution for rotor excited system

Steady state solution to

\[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2y}{dt^2} \]

\[ \omega_n = \sqrt{\frac{k}{m + m_0}} \quad \zeta = \frac{c}{2\sqrt{k(m + m_0)}} \quad K = \frac{m_0}{m + m_0} \]

\[ x_{ss}(t) = X_0 \sin(\omega t + \phi) \]

\[ X_0 = KY_0 M(\omega, \omega_n, \zeta) \quad M = \frac{\omega^2 / \omega_n^2}{\left(1 - \omega^2 / \omega_n^2\right)^2 + (2\zeta\omega / \omega_n)^2} \]

\[ \phi = \tan^{-1} \frac{-2\zeta\omega / \omega_n}{1 - \omega^2 / \omega_n^2} \]

\[ y = Y_0 \sin \omega t \]

\[ \theta = \omega t \]
Understanding solution

\[ x_p = X_0 \sin(\omega t + \phi) \]

\[ X_0 = \frac{k}{m_0} m \]

\[ K = \frac{m_0}{m + m_0} \]

\[ M = \frac{\omega^2/\omega_n^2}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta \omega/\omega_n)^2}} \]

1. \( m \approx \omega^2/\omega_n^2 \Rightarrow X_0 = (m_0 \omega^2/k) Y_0 \)

2. \( M \approx \frac{1}{2\zeta} \Rightarrow X_0 = \left[ \frac{m_0}{m + m_0} \right] \left[ \frac{Y_0}{2\zeta} \right] \) (resonance)

3. \( M \approx 1 \Rightarrow X_0 = \left[ \frac{m_0}{m + m_0} \right] Y_0 \)
5.6.11 Example: A motor with total mass \( m + m_0 = 50 \text{kg} \) has rotating internal mass of \( m_0 = 1 \text{kg} \) that rotates on a shaft with eccentricity \( Y_0 = 1 \text{mm} \) at angular rate \( \omega = 100 \text{rad/s} \). The engine is mounted on vibration isolation pads with effective stiffness \( k = 500000 \text{N/m} \) and a dashpot coefficient \( c = 250 \text{ Ns/m} \). The system is found to have a severe vibration problem. Which of the following changes will reduce the vibration amplitude?

(a) Increase the pad stiffness \( k \)
(b) Decrease the pad stiffness \( k \)
(c) Increase the speed of the motor
(d) Decrease the dashpot coefficient \( c \)

\[
W_n = \sqrt{\frac{k}{(m+m_0)}} = 100 \text{ rad/s} \quad \zeta = \frac{c}{2\sqrt{k(m+m_0)}} = 0.025
\]

Hence \( \omega / W_n = 1 \) \( \Rightarrow \) resonance!

(a) Increases \( W_n \) \( \Rightarrow \) reduces amplitude
(b) Decreases \( W_n \) \( \Rightarrow \) reduces amplitude
(c) Increases \( \omega \) \( \Rightarrow \)
(d) Decreases \( \zeta \) \( \Rightarrow \) increases amplitude

\[\Rightarrow \text{ (a), (b), (c) reduce amplitude} \]
5.6.12 Example: An unbalanced wind turbine is idealized as a rotor excited spring-mass system. The mass $m$ represents the tower, and $m_0$ represents the combined mass of the three rotor blades. The rotor is ‘unbalanced’ because its center of mass is a distance $Y_0$ away from the axle. The total mass of the turbine is 25000kg, the spring stiffness is 4100 kN/m and the dashpot coefficient is 128 kN·s/m.

The figure shows the measured displacement of the system during operation. The blades have a radius of 40m. Assuming that the rotor can be balanced by adding mass to the tip of one blade, estimate the mass that must be added to balance the rotor.

**Approach:**
1. Balance rotor by moving COM to $r = 0$
2. Recall $r = (\sum m_i r_i) / (\sum m_i) \Rightarrow m_0 Y_0 - m^* R = 0$
3. To find $m^*$ we need to know $m_0 Y_0$
4. Recall $X_0 = E Y_0 M = [m_0 Y_0 / (m + m_0)] M$
   \[
   \Rightarrow \text{Find } m_0 Y_0 \text{ using } X_0, M, (m + m_0) \text{ (all given)}
   \]
5.6.12 Example: An unbalanced wind turbine is idealized as a rotor excited spring-mass system. The mass \( m \) represents the tower, and \( m_0 \) represents the combined mass of the three rotor blades. The rotor is ‘unbalanced’ because its center of mass is a distance \( Y_0 \) away from the axle. The total mass of the turbine is 25000kg, the spring stiffness is 4100 kN/m and the dashpot coefficient is 128 kNs/m.

\[
W_n = \sqrt{\frac{k}{(m+m_0)}} = 12.8 \text{ rad/s}
\]

\[
\zeta = \frac{c}{2\zeta k(m+m_0)} = 0.2
\]

\[
X_0 = \frac{K}{M} Y_0 = \frac{m_0}{Y_0} \frac{W^2/W_n^2}{(m+m_0) \sqrt{(1-W^2/W_n^2)^2 + (25W/W_n)^2}}
\]

\[
W = 1 \text{ rad/s} \quad X_0 = 6 \times 10^{-3} \text{ m}
\]

\[
\Rightarrow Y_0 m_0 = 25 \times 10^3 \text{ kg m}
\]

To balance turbine add mass \( m^* \) at radius

\[
R = 40 \text{ m} \quad Y_0 m_0 + m^* (-R) = 0
\]

\[
\Rightarrow m^* = \frac{Y_0 m_0}{R} = 625 \text{ kg}
\]
5.6.13 Example: The figure shows an ‘anti-resonant’ vibration isolation system.
(a) Find the equation of motion relating \( x(t) \) to \( y(t) \).
Assume \( \theta << 1 \) and neglect gravity
(b) Plot the ‘transmissibility’ of the isolator as a function of frequency for
\[
k = 20 \text{kN/m} \quad m_1 = 1 \text{kg}, \quad m_2 = 10 \text{grams}, \quad c = 20 \text{Ns/m}
\]
\[
L_2 / L_1 = 10
\]

Equation of motion

Geometry:
\[
z = y + \frac{L_2}{2} + L_2 \sin \theta \quad x = y + \frac{L_1}{2} + L_2 \theta
\]
\[
x = y + L - L_1 \sin \theta \quad x = y + L - L_1 \theta
\]
\[
r = L_2 \cos \theta \approx L_2
\]

\[
\Rightarrow \frac{d^2 z}{dt^2} = \frac{d^2 y}{dt^2} + L_2 \frac{d^2 \theta}{dt^2}
\]
\[
\Rightarrow \frac{d^2 \theta}{dt^2} = \frac{1}{L_1} \frac{d^2 (y - x)}{dt^2}
\]
\[
\Rightarrow \frac{d^2 \theta}{dt^2} = \frac{1}{L_1} \frac{d^2 (y - x)}{dt^2} \Rightarrow \frac{d^2 z}{dt^2} = -L_2 \frac{d^2 x}{dt^2} + \left(1 + \frac{L_2}{L_1}\right) \frac{d^2 y}{dt^2}
\]
\[ F_s = k (x - y - L) \]

\[ F_d = c \frac{d}{dt} (x - y) \]

\[ F_y = m_1 a \] for mass \( m_1 \)

\[ m_1 \frac{d^2 x}{dt^2} = -F_s - F_d - T \]  \( (2) \)

\[ F_y = m_2 a \] for \( m_2 \) \[ \Rightarrow m_2 \frac{d^2 z}{dt^2} = R_y + T \] \( (3) \)

\[ F_x = m_2 a \] for \( m_2 \) \[ \Rightarrow m_2 \frac{d^2 r}{dt^2} = R_x \] \[ \Rightarrow R_x = 0 \]

\[ \leq M_{\text{com}} = 0 \] for \( m_2 \)

\[ \Rightarrow -T (L_1 + L_2) \cos \theta - R_y L_2 \cos \theta + R_x L_1 \sin \theta = 0 \]

\[ \Rightarrow R_y = -T (L_1 + L_2) / L_2 \]

\[ (3) \Rightarrow m_2 \frac{d^2 z}{dt^2} = -T \frac{L_1}{L_2} \] \( (4) \)
\[(1), (2), (4) \Rightarrow \]
\[
\left( \frac{m_1 + m_2}{K} \right) \frac{d^2x}{dt^2} + \frac{C}{K} \frac{dx}{dt} + x = \frac{L}{K} + m_2 \left( \frac{L_2}{L_4} \right) \frac{d^2y}{dt^2} + C \frac{dy}{dt} + y
\]

(b) Substitute numbers
\[
\frac{1}{10^4} \frac{d^2x}{dt^2} + \frac{1}{10^3} \frac{dx}{dt} + x = \frac{L}{10^4} + 0.55 \frac{d^2y}{dt^2} + \frac{1}{10^3} \frac{dy}{dt} + y
\]

List of standard ODEs for vibration problems

Case I
\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + x = C
\]

Compare coefficients:
\[
\frac{1}{\omega_n^2} = \frac{1}{10^4} \quad \frac{2\zeta}{\omega_n} = \frac{1}{10^3} \quad \frac{2\zeta^2}{\omega_n^2} = \frac{55}{10^4}
\]

\[\Rightarrow \quad \omega_n = 100 \text{ rad/s} \]

Case II
\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} - x = -C
\]

\[\zeta = 0.05\]

Case III
\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C
\]

\[\lambda = 0.742\]

Case IV
\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + KF(t) \quad \text{with} \quad F(t) = F_0 \sin \omega t
\]

\[K = 1\]

Case V
\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left( y + \frac{2\zeta}{\omega_n} \frac{dy}{dt} \right) \quad \text{with} \quad y(t) = Y_0 \sin \omega t
\]

Case VI
\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C - K \frac{d^2y}{dt^2} \quad \text{with} \quad y(t) = Y_0 \sin \omega t
\]

Case VII
\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = K \left( \frac{\lambda^2}{\omega_n^2} \frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy}{dt} + y \right) \quad \text{with} \quad y(t) = Y_0 \sin \omega t
\]
Steady-State Solution to Case VII (From pdf on website)

Equation

\[ \frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dx}{dt} + x = C + K \left( \frac{\lambda^2}{\omega_n^2} \frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_n} \frac{dy}{dt} + y \right) \]

\[ y(t) = Y_0 \sin(\omega t) \]

\[ X_0 = KY_0 M(\omega / \omega_n, \zeta, \lambda) \]

\[ M(\omega / \omega_n, \zeta, \lambda) = \left\{ \frac{(1-\lambda^2\omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2}{(1-\omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2} \right\}^{1/2} \]

\[ \phi = \cos^{-1} \left\{ \frac{1-\lambda^2\omega^2 / \omega_n^2}{(1-\lambda^2\omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2} \right\}^{1/2} - \cos^{-1} \left\{ \frac{1-\omega^2 / \omega_n^2}{(1-\omega^2 / \omega_n^2)^2 + (2\zeta\omega / \omega_n)^2} \right\}^{1/2} \]

\[ (-\pi < \phi < 0) \]

"Transmissibility" = \( X_0 / Y_0 = M \)

Plot graph w/ MATLAB

Note antiresonance!