Course Outline

1. MATLAB tutorial
2. Motion of systems that can be idealized as particles
   - Description of motion, coordinate systems; Newton’s laws;
   - Calculating forces required to induce prescribed motion;
   - Deriving and solving equations of motion
3. Conservation laws for systems of particles
   - Work, power and energy; ← You are here!
   - Linear impulse and momentum
   - Angular momentum
4. Vibrations
   - Characteristics of vibrations; vibration of free 1 DOF systems
   - Vibration of damped 1 DOF systems
   - Forced Vibrations
5. Motion of systems that can be idealized as rigid bodies
   - Description of rotational motion
   - Kinematics; gears, pulleys and the rolling wheel
   - Inertial properties of rigid bodies; momentum and energy
   - Dynamics of rigid bodies
2. Energy methods
   - Definition of rate of work (power) developed by a force
   - Definition of total work done by a force
   - Definition of kinetic energy
   - Power/work/kinetic energy relation for a single particle
2) Conservation laws for systems of particles

2.1 Work - Power - Energy relations for particles

Definitions

**Power (rate of work) of a force**

\[ P = F \cdot v \quad \text{"Watts"} \quad \text{kg m}^2/\text{s}^3 \]

**Work Done by a force during time** 

\[ W = \int_{t_0}^{t} P(t) \, dt = \int_{t_0}^{t} F \cdot v \, dt \]

\[ W = \int_{r_0}^{r_1} F \cdot dr \quad \text{"Joules"} \quad \text{kg m}^2/\text{s}^2 \]
Inverse relation

Given \( W(x, y, z) \)

Then \( F = \nabla W = \frac{\partial W}{\partial x} \mathbf{i} + \frac{\partial W}{\partial y} \mathbf{j} + \frac{\partial W}{\partial z} \mathbf{k} \)

Kinetic Energy

\[
T = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m \mathbf{v} \cdot \mathbf{v} = \frac{1}{2} m \mathbf{v}^2 = \frac{1}{2} m \left( v_x^2 + v_y^2 + v_z^2 \right) \quad \text{(also Joules)}
\]
Work - Power - Energy relations for a single particle

Let \( F \) be total force acting on a particle

\[ P = \dot{V} \quad \text{(Power)} \]

\[ T = \frac{1}{2} m |V|^2 \quad \text{(KE)} \]

or

\[ W = T_1 - T_0 \]

\[ T_i = KE \text{ at time } t_i \]

\[ T_0 = KE \text{ at time } t_0 \]

\[ W = \int_{t_0}^{t_1} P(t) \, dt \quad \text{work} \]
Proof: \[ F = ma = m \frac{dv}{dt} \]

\[ \Rightarrow F \cdot v = m \frac{dv}{dt} \cdot v \]

P

Note: \[
\frac{d}{dt} \left( \frac{1}{2} m \cdot v \cdot v \right) = \frac{1}{2} m \left( \frac{dv}{dt} \cdot v + v \cdot \frac{dv}{dt} \right) = m \frac{dv}{dt} \cdot v
\]

\[ \Rightarrow p = \frac{d}{dt} \left( \frac{1}{2} m v^2 \right) \]

Finally: \[
\int_{t_0}^{t_1} p \, dt = \int_{t_0}^{t_1} \frac{d}{dt} \left( \frac{1}{2} m v^2 \right) \, dt \Rightarrow \bar{W} = \bar{F} = \bar{T}_1 - \bar{T}_0
\]
Example 1: An aircraft with mass 15000 kg flying at 200 knots (102 m/s) climbs at 1000 ft/min (5.1 m/s). Calculate the rate of work done on the aircraft by gravity.

Definition \( P = F \cdot v = -mg_f \cdot (v_x i + v_y j) \)

\[ = -mg \cdot v_y = 750 \text{ kw} \]
Example 2: An external force $F(x)$ stretches a spring with stiffness $k$ and unstretched length $L_0$ from length $L_1$ to length $L_2$. Calculate the work done by the force.

\[ W = \int_{L_0}^{L_2} F \cdot dx = \int_{L_1}^{L_2} k \left( x - L_0 \right) dx \]

\textit{Spring formula}

\[ W = \int_{L_1}^{L_2} k \left( x - L_0 \right) dx \]

\[ = \left[ \frac{1}{2} (x - L_0)^2 \right]_{L_1}^{L_2} = \frac{1}{2} k \left\{ (L_2 - L_0)^2 - (L_1 - L_0)^2 \right\} \]
Example 3: The work per unit area required to separate two atomic planes in a crystal by a distance \( x \) can be approximated by

\[
W(x) = -E_0 \left( 1 + \frac{x}{d} \right) \exp \left( -\frac{x}{d} \right)
\]

1. Calculate the force of attraction (per unit area) between the planes as a function of \( x \)
2. Calculate the stiffness of the bond between the planes

\[
F = \frac{\partial W}{\partial x} = \frac{\partial W}{\partial y} = \frac{\partial W}{\partial z}
\]

\[
F = \frac{\partial W}{\partial x} = \left[ -\frac{E_0}{d^2} \exp \left( -\frac{x}{d} \right) + \frac{E_0}{d} \left( 1 + \frac{x}{d} \right) \exp \left( -\frac{x}{d} \right) \right] \frac{x}{d} \exp \left( -\frac{x}{d} \right)
\]
For small forces interface behaves like a spring. Stiffness = slope of force-$v$-x curve at $F=0$.

$$k = \frac{dF}{dx} \bigg|_{x=0}$$

$$\frac{dF}{dx} = -\frac{E_0}{d^2} \exp\left(-\frac{x}{d}\right) - \frac{E_0x}{d^3} \exp\left(-\frac{x}{d}\right)$$

$$\Rightarrow k = \frac{E_0}{d^2}$$
Example 4: Estimate time for Ferrari California to reach 100km/hr

Specifications
- Mean engine power 200 KW
- Mass 1735 kg
- Frontal area 4 m²
- Drag coefficient 0.32
- Drag force $F_D = \frac{1}{2} \rho C_D A V^2$

Approach: Power - KE formula for a particle

\[ P = \frac{dT}{dt} = \frac{d}{dt} \left( \frac{1}{2} m V^2 \right) = m V \frac{dV}{dt} \quad (1) \]

Engine & drag do work on car

\[ P = P_{\text{engine}} + P_{\text{drag}} \quad (2) \]
Drag force \( \mathbf{F}_D = -\frac{1}{2} \rho C_D A \mathbf{V} \mathbf{V} \) \( \text{unit vector parallel to motion} \)

\[
\mathbf{P}_{\text{drag}} = \mathbf{F}_D \cdot \mathbf{V} = -\frac{1}{2} \rho C_D A V^2 \frac{\mathbf{V} \cdot \mathbf{V}}{V} = -\frac{1}{2} \rho C_D A V^3 \quad (3)
\]

Combine (1), (2), (3)

\[
P_{\text{engine}} - \frac{1}{2} \rho C_D A V^3 = m \frac{V}{dV} dt
\]

\[
\Rightarrow \quad \int_{V_{\text{max}}}^{V_0} \frac{m V}{dV} dt = \int_0^t dt = t
\]

\[
\int_0^{P_{\text{engine}} - \frac{1}{2} \rho C_D A V^3} dt = \int_0^t dt = t
\]
Use MATLAB to do the integral on the left

```matlab
syms V t
Pe = 200000; m = 1735; A = 4; cD = 0.32; rho = 1.2
Vmax = 100000/3600; \% 100 km/hr in m/s
eval(int(m*V/(Pe-rho*cD*A*V^3/2),[0,Vmax]))
```

ans = 3.4631
Example: Estimate length of crumple zone required to provide protection from 30mph crash
Assume crumple zone exerts a constant force
Constraint – max accel must not exceed 8g

**Approach:**

1. Find $F$ with $F = ma$
2. Work–KE relation to find $d$
3. $F = ma$, $F_i = max_i$

Since $ax < 8g$ (given)

$\Rightarrow F < 8mg$
(2) Work-energy relation

\[ W = T_1 - T_0 \]

\[ T_1 = 0 \quad \text{(car at rest)} \]

\[ T_0 = \frac{1}{2} m V_0^2 \quad \text{(before impact)} \]

\[ W = \int_0^0 F \cdot dx = \int_0^0 F \cdot dx = \int_0^0 F \cdot dx \]

\[ \Rightarrow W = [F \cdot x]^0_0 = -Fd \]

\[ -Fd = 0 - \frac{1}{2} m V_0^2 \quad \Rightarrow \quad d = \frac{1}{2} m \frac{V_0^2}{F} \]

\[ \Rightarrow d > \frac{V_0^2}{(16g)} = 1.2 \text{ m} \]

(since \( F < 8mg \))