Course Outline

1. MATLAB tutorial
2. Motion of systems that can be idealized as particles
   - Description of motion, coordinate systems; Newton’s laws;
   - Calculating forces required to induce prescribed motion;
   - Deriving and solving equations of motion
3. Conservation laws for systems of particles
   - Work, power and energy; ← You are here!
   - Linear impulse and momentum
   - Angular momentum
4. Vibrations
   - Characteristics of vibrations; vibration of free 1 DOF systems
   - Vibration of damped 1 DOF systems
   - Forced Vibrations
5. Motion of systems that can be idealized as rigid bodies
   - Description of rotational motion
   - Kinematics; gears, pulleys and the rolling wheel
   - Inertial properties of rigid bodies; momentum and energy
   - Dynamics of rigid bodies
Review of concepts

Rate of work done by a force (power developed by force)

\[ P = \mathbf{F} \cdot \mathbf{v} \]

Total work done by a force

\[ W = \int_{0}^{t_1} \mathbf{F} \cdot \mathbf{v} \, dt \quad W = \int_{r_0}^{r_1} \mathbf{F} \cdot d\mathbf{r} \]

Kinetic energy

\[ T = \frac{1}{2} m |\mathbf{v}|^2 = \frac{1}{2} m \left( v_x^2 + v_y^2 + v_z^2 \right) \]

Power-kinetic energy relation

\[ P = \frac{dT}{dt} \]

Work-kinetic energy relation

\[ W = \int_{r_0}^{r_1} \mathbf{F} \cdot d\mathbf{r} = T - T_0 \]
Topics for today’s class

Energy methods

- Conservative, non-conservative and workless forces
- Potential energy of a conservative force
- Conservative systems
- Energy conservation equation for conservative systems
2.3 Conservative Forces and Potential Energy

Recall work done by a force

\[ W = \int_{r_0}^{r_1} F(r) \cdot dr \]

**Definition:** \( F \) is conservative \( \iff \) \( W \) is equal for all paths from \( r_0 \) to \( r_1 \)

**Examples:**
- Conservative
  - Gravity
  - Electrostatic forces
  - Inter-molecular forces
  - Forces exerted by springs

- Non-Conservative
  - Friction
  - Air resistance

- Workless
  - Lift force
  - Reaction forces
  - Lorenz force
Definition: Potential Energy of conservative force

\[ PE : U(E) = -\int_{r_0}^{r_1} \mathbf{F} \cdot d\mathbf{r} + C \]

We can use any path.

So, C are arbitrary - choose to make \( U \) simple.

Inverse form

\[ \mathbf{F} = -\nabla U = -\left\{ \frac{\partial U}{\partial x} \mathbf{i} + \frac{\partial U}{\partial y} \mathbf{j} + \frac{\partial U}{\partial z} \mathbf{k} \right\} \]
Example: Calculate potential energy of gravity

\[ F = -\frac{GMm}{r^2} \hat{e}_r, \]

\[ G = 6.674 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} \]

Formula: \[ U = -\int_{r_0}^{r} \mathbf{F} \cdot d\mathbf{r} \]

Use \[ \hat{e}_r, \hat{e}_\theta, \hat{e}_z \] basis

\[
\begin{align*}
U &= -\int_{r_0}^{r} -\frac{GMm}{r^2} \hat{e}_r \cdot dr \hat{e}_r + C = \int_{r_0}^{r} \frac{GMm}{r^2} dr + C \\
U &= \left[ -\frac{GMm}{r} \right]_{r_0}^{r} + C = -\frac{GMm}{r} + \frac{GMm}{r_0} + C \\\n\text{Choose } C = -\frac{GMm}{r_0} \Rightarrow \quad U &= -\frac{GMm}{r} \]
\]
Formulas for potential energies of common forces

<table>
<thead>
<tr>
<th>Type of force</th>
<th>Force vector</th>
<th>Potential energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity acting on a particle near earth's surface</td>
<td>$F = -mgj$</td>
<td>$U = mgy$</td>
</tr>
<tr>
<td>Gravitational force exerted on mass $m$ by mass $M$ at the origin</td>
<td>$F = -\frac{GMm}{r^2} \mathbf{r}$</td>
<td>$U = -\frac{GMm}{r}$</td>
</tr>
<tr>
<td>Force exerted by a spring with stiffness $k$ and unstretched length $L_0$</td>
<td>$F = -k(r - L_0) \frac{\mathbf{r}}{r}$</td>
<td>$U = \frac{1}{2} k(r - L_0)^2$</td>
</tr>
<tr>
<td>Force acting between two charged particles</td>
<td>$F = \frac{Q_1 Q_2}{4\pi \varepsilon r^2} \mathbf{r}$</td>
<td>$U = \frac{Q_1 Q_2}{4\pi \varepsilon r}$</td>
</tr>
<tr>
<td>Force exerted by one molecule of a noble gas (e.g. He, Ar, etc) on another (Lennard Jones potential). $a$ is the equilibrium spacing between molecules, and $E$ is the energy of the bond.</td>
<td>$F = 12 \frac{E}{a} \left[ \left( \frac{a}{r} \right)^{13} - 2 \left( \frac{a}{r} \right)^{7} \right] \frac{\mathbf{r}}{r}$</td>
<td>$U = E \left[ \left( \frac{a}{r} \right)^{12} - 2 \left( \frac{a}{r} \right)^{6} \right]$</td>
</tr>
</tbody>
</table>
Example: The potential energy of two neighboring Cheerios floating in milk is

\[ U \approx E_0 \log \left( \frac{r}{L_0} \right) \quad r = \sqrt{x^2 + y^2} \]

where \( E_0, L_0 \) are constants

Find a formula for the force acting between them.

\[ F = -\nabla U \]

\[ = -\frac{\partial U}{\partial x} \mathbf{i} - \frac{\partial U}{\partial y} \mathbf{j} \]

Use chain rule

\[ F = -\frac{\partial U}{\partial r} \frac{\partial r}{\partial x} \mathbf{i} - \frac{\partial U}{\partial r} \frac{\partial r}{\partial y} \mathbf{j} \]

\[ \frac{\partial r}{\partial x} = \frac{1}{2} \frac{1}{\sqrt{x^2 + y^2}} \cdot 2x = \frac{x}{r} \]

\[ \frac{\partial r}{\partial y} = \frac{y}{r} \]

\[ \Rightarrow F = - \frac{E_0}{r} \left( \frac{x}{r} \mathbf{i} + \frac{y}{r} \mathbf{j} \right) \]

Magnitude. Unit Vector: Direction
In general if PE has form $U(r)$ then $\frac{dU}{dr}$ gives force magnitude.

$\frac{dU}{dr} > 0 \Rightarrow F$ is attractive.

$\frac{dU}{dr} < 0 \Rightarrow$ repulsive.
2.4 Energy Equation for a conservative system of particles

**Definition:** System is conservative if we can calculate a potential energy for all internal forces.

- $\mathbf{R}_{ij}$: Force exerted on particle $i$ by particle $j$
- $\mathbf{F}_{ext}^i$: External force on particle $i$
- $\mathbf{r}_i$: Position of particle $i$
- $\mathbf{v}_i$: Velocity of particle $i$

Let $U_{ij}$ be potential energy of force between particles $i$ and $j$.

$$\mathbf{R}_{ij} = - \frac{\partial U_{ij}}{\partial \mathbf{r}_i}$$
Definitions:

1. Total Potential Energy \( U^{\text{total}} = \sum_{\text{interm}} U_{ij} \)

2. Total KE of system \( T^{\text{total}} = \sum \frac{1}{2} m_i v_i^2 \)

3. Total power of external forces \( P_{\text{ext}} = \sum_{\text{exter}} F_i \cdot v_i \)
Energy Equation

Version 1

\[ P^{ext} = \frac{d}{dt} \left\{ U^{TOT} + T^{TOT} \right\} \]

Version 2

\[ \Delta W^{ext} = \left( U_1^{TOT} + T_1^{TOT} \right) - \left( U_0^{TOT} + T_0^{TOT} \right) \]

Special case \( \Delta W^{ext} = 0 \)

\[ U_1^{TOT} + T_1^{TOT} = U_0^{TOT} + T_0^{TOT} \]

"Total Energy is conserved"
Proof of the energy equation for a conservative system

1. Power-energy relation for a single particle
\[ \left( F_i^{\text{ext}} + \sum_j R_{ij} \right) \cdot v_i = \frac{dT_i}{dt} \]

2. Sum over all particles
\[ \left( \sum_i F_i^{\text{ext}} + \sum_i \sum_j R_{ij} \right) \cdot v_i = \sum_i \frac{dT_i}{dt} \]

3. Recall \( R_{ij} = -\frac{\partial U_{ij}}{\partial r_i} \) and note \( v_i = \frac{\partial r_i}{\partial t} \)
\[ \Rightarrow P^{\text{ext}} + \sum_i \sum_j -\frac{\partial U_{ij}}{\partial r_i} \cdot \frac{\partial r_i}{\partial t} = \frac{dT^{\text{TOT}}}{dt} \]

4. Note \( \sum_i \sum_j -\frac{\partial U_{ij}}{\partial r_i} \cdot \frac{\partial r_i}{\partial t} = -\frac{\partial U^{\text{TOT}}}{\partial t} \)

Hence \[ P^{\text{ext}} = \frac{d}{dt} \left( U^{\text{TOT}} + T^{\text{TOT}} \right) \]
**Example:** A bungee jumper is idealized as a spring-mass system.

The mass starts from rest with \( v = L_0 \)

Calculate the drop distance \( y \) and the maximum acceleration of the mass.

---

**Procedure**

1. Choose system
2. Identify state \((0)\) & \((1)\)
3. Use energy equation

---

**Here:**

1. **System** is jumper, spring, & earth
2. State \((0)\) is just before drop
   \((1)\) is at max deflection
Notes: (1) No ext forces $\Rightarrow$ $\Delta W^{\text{ext}} = 0$

(2) Jumper is at rest in states (0) & (1)

Formulas: $U^{\text{spring}} = \frac{1}{2} k (z - z_0)^2$, $U^{\text{grav}} = mgh$

Hence $\Delta W^{\text{ext}} = 0$

$U_0^{\text{tot}} = \frac{1}{2} k (z_0 - z_0)^2 + mg (-z_0)$

$T_0^{\text{tot}} = 0$

$U_1^{\text{tot}} = \frac{1}{2} k (y - z_0)^2 + mg (-y)$

$T_1^{\text{tot}} = 0$
Energy Eq.

\[ \Delta W^\text{ext} = U_1^\text{tot} + T_1^\text{tot} - (U_0^\text{tot} + T_0^\text{tot}) \]

\[ 0 = \frac{1}{2} k (y - L_0)^2 - mg y - (-mg L_0) \]

\[ \Rightarrow \frac{1}{2} k (y - L_0)^2 = mg (y - L_0) \]

\[ y = L_0 + \frac{2mg}{k} \]

Max Accel:

\[ F = ma \quad \text{at bottom of jump} \]

\[ ma = F_s - mg \]

\[ = k (y - L_0) - mg \]

\[ = mg \quad \Rightarrow \quad a = g \]