3.3 Vibration of systems with several DOF

1. Vibration is not generally harmonic
2. There exist special initial displacements that cause harmonic motion; each has a different frequency
3. Special displacements are called "Vibration modes" or "Normal modes"
4. Special frequencies are called "Natural frequencies"; natural frequencies are close to resonant frequencies
Natural Frequencies and Mode Shapes

General system does not always vibrate harmonically.

All unforced undamped systems vibrate harmonically at special frequencies, called Natural Frequencies of the system.

The system will vibrate harmonically if it is released from rest with a special set of initial displacements, called Mode Shapes or Vibration Modes.
3.4 Counting DOF and vibration modes

$\# \text{DOF} = \# \text{coordinates needed to describe motion}$

$\# \text{Vibration modes} = \# \text{of independent init conditions that cause harmonic motion}$

**Formal approach:** Let

$\# \text{rigid bodies} = r \quad \# \text{particles} = p \quad \# \text{constraints} = c$

Then:

- 3D system $\# \text{DOF} = 6r + 3p - c$
- 2D system $\# \text{DOF} = 3r + 2p - c$

$\# \text{Vibration modes} = \# \text{DOF} - "\# \text{Rigid Body Modes}"

Rigid Body modes = non-vibrational motion

Either translation @ const speed

Or rotation

Max 6 in 3D or 3 in 2D
Counting Constraint

Pinned joint
(5 constraints – prevents all motion, and prevents rotation about two axes)

Rigid (massless) link
If the link has mass, it should be represented as a rigid body.
1 constraint (prevents relative motion parallel to link)

Nonconformal contact
(two bodies meet at a point)
No friction or slipping: 1 constraint (prevents interpenetration)

Sticking friction
2 constraints (prevents relative motion)

Conformal contact
(two rigid bodies meet along a line)
No friction or slipping: 2 constraints (prevents interpenetration and rotation)

Sticking friction
3 constraints (prevents relative motion)

Pinned joint
(generally only applied to a rigid body, as it would stop a particle moving completely)
2 constraints (prevents motion horizontally and vertically)
Example

Here \( r = 3 \) rigid bodies \( p = 0 \)

Each pin joint prevents relative motion in 2 directions \( \Rightarrow C = 2 \)

\( \Rightarrow C = 4 \)

HD OF \( = 3 \times 3 - 4 = 5 \)

\((3r - c) \)

\((x_c, y_c, 0, 0, O_1, O_2, O_3)\)

In the air we have 3 rigid body modes

(2 translation & 1 rotation)

\( \Rightarrow \# \text{ Vibe Modes} = 5 - 3 = 2 \)

(a) **Model of a hopping robot** (motion is confined to the \( x, y \) plane. Consider the robot in the air only)
Example: Methane

5 particles (atoms)

0 rigid bodies or constraints

# DOF = 3p = 15

# Vibration mode = # DOF - # rigid body modes

Molecule has 6 "rigid body" modes

# Vibration modes = 15 - 6 = 9
$v_1$: Symmetric C-H Stretch

$v_2$: Doubly Degenerate Bend

$v_3$: Triply Degenerate Antisymmetric C-H Stretch

$v_4$: Triply Degenerate Bend
3.5) Methods for calculating natural frequencies

**Combining Springs**

Replace many springs by $k_{eff}$

Formula: $\omega_n = \sqrt{\frac{k_{eff}}{m}}$

**Parallel Springs**

(Forces add)

- Parallel springs
  
  $F = F_{s1} + F_{s2}$
  
  $F = k_1 (L-L_0) + k_2 (L-L_0)$
  
  $F = (k_1 + k_2)(L-L_0)$
Effective spring: \[ F = k_{\text{eff}} (L - L_0) \]

Compare \(\Rightarrow\) \[ k_{\text{eff}} = k_1 + k_2 \]

**Series springs**

- For series springs
  \[ F \quad L = L_1 + L_2 \]
  \[ F = k_1 (L_1 - L_{01}) \]
  \[ F = k_2 (L_2 - L_{02}) \]

\[ \Rightarrow L = L_{01} + L_{02} + \frac{F}{k_1} + \frac{F}{k_2} \]

- Effective spring
  \[ L = \frac{1}{k_{\text{eff}}} F + L_{\text{eff}} \]

\[ \frac{1}{k_{\text{eff}}} = \frac{1}{k_1} + \frac{1}{k_2} \]

\[ L_{\text{eff}} = L_{01} + L_{02} \]
Example

Are these series or parallel?

Visualize mass displaced by distance $x$

Forces add $\Rightarrow$ springs are parallel

\[ R_{\text{eff}} = k_1 + k_2 \]

\[ \omega_n = \sqrt{\frac{R_{\text{eff}}}{m}} = \sqrt{\frac{k_1 + k_2}{m}} \]
Finding EOM using energy (conservative systems)

Trick: Conservative \( \Rightarrow \) KE + PE = const

\[ \Rightarrow \frac{d}{dt}(KE + PE) = \frac{d}{dt}(\text{const}) = 0 \]

Here \( KE = \frac{1}{2} m (\frac{ds}{dt})^2 \)

\( PE = \frac{1}{2} k (s - l_0)^2 \)

\[ \frac{d}{dt} \left\{ \frac{1}{2} m (\frac{ds}{dt})^2 + \frac{1}{2} k (s - l_0)^2 \right\} = m \frac{ds}{dt} \frac{d^2 s}{dt^2} + k (s - l_0) \frac{ds}{dt} = 0 \]

\[ \Rightarrow m \frac{d^2 s}{dt^2} + k (s - l_0) = 0 \] (same as before)