Topics for today's class

1. Tricks for calculating natural frequency of a 1DOF undamped system
   - Calculating natural frequencies for NONLINEAR systems
     http://www.minusk.com/

2. Damped Vibrations
   - The dashpot (damper); dashpot coefficient
   - EOM for damped vibrating system
   - Standard form of EOM for damped vibrating system – natural frequency, damping factor (damping ratio/damping coefficient)
   - Solutions to EOM – underdamped, critically damped, overdamped
Review

Tricks for calculating natural frequencies of 1DOF undamped systems

Using energy conservation to find EOM

\[ KE + PE = \frac{1}{2} m \left( \frac{ds}{dt} \right)^2 + \frac{1}{2} k (s - L_0)^2 = \text{const} \]

\[ \Rightarrow \frac{d}{dt} (KE + PE) = m \left( \frac{ds}{dt} \right) \frac{d^2 s}{dt^2} + k (s - L_0) \frac{ds}{dt} = 0 \]

\[ \Rightarrow m \frac{d^2 s}{dt^2} + ks = kL_0 \]
Example using energy method to find EOM

Find EOM for pendulum

\[ KE = \frac{1}{2} m V^2 \quad \text{Circular Motion} \quad V = L \frac{d\theta}{dt} \]

\[ KE = \frac{1}{2} mL^2 \left( \frac{d\theta}{dt} \right)^2 \]

\[ PE = mgh = -mgL\cos\theta \]

\[ \frac{d}{dt} \left\{ PE + KE \right\} = mL^2 \frac{d^2\theta}{dt^2} + mgL\sin\theta \frac{d\theta}{dt} = 0 \]

\[ L^2 m \frac{d^2\theta}{dt^2} + mgL \sin\theta = 0 \]

"Nonlinear" - EOM is not in tables

How to find \( a_n \)?
2.6 Finding natural freq for nonlinear systems

General Problem: \( A \frac{d^2y}{dt^2} + f(y) = 0 \)

- \( f(y) \) satisfies \( f(0) = 0 \)

Approach: Assume small oscillations \( y << 1 \)

Expand \( f(y) \) as Taylor series:

\[
f(y) \approx f(0) + f'(0) y + \frac{1}{2} f''(0) y^2 + \ldots
\]

\[
= f(0) + f'(0) y + \frac{1}{2} f''(0) y^2
\]

Neglect all terms

EOM: \( A \frac{d^2y}{dt^2} + f'(0) y = 0 \)
Rearrange: \( \frac{1}{\omega_n^2} \left\{ \frac{A}{f'(0)} + y \right\} \frac{d^2y}{dt^2} + y = 0 \)

We have solution for: \( \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + x = 0 \)

\[ \omega_n = \sqrt{\frac{f'(0)}{A}} \]

For pendulum: \( f(y) = \sin(\theta) \approx \sin(\theta) + \cos(\theta) \theta \)

\[ \frac{1}{\omega_n^2} \left( \frac{L}{g} \right) \frac{d^2\theta}{dt^2} + \theta = 0 \]

\[ \omega_n = \sqrt{\frac{g}{L}} \]
Example: calculate natural frequency of the ‘minus k’ vibration isolation system

\[ \sqrt{d^2 + y^2} \]

**Approach:** Find EOM (energy)

"Linearize" EOM using Taylor approx

Read off \( \omega_n \)

Neglect gravity (no effect on \( \omega_n \))

\[ k\overline{E} = \frac{1}{2} m \left( \frac{dy}{dt} \right)^2 \]
\[ PE = \frac{1}{2} k_1 y^2 + \frac{1}{2} k_2 x^2 \left( \sqrt{d^2 + y^2} - L_0 \right)^2 \]

\[
\frac{d}{dt} \left\{ KE + PE \right\} = m \frac{dy}{dt} \frac{d^2 y}{dt^2} + k_1 y \frac{dy}{dt} + \]

\[
+ 2 k_2 \left( \sqrt{d^2 + y^2} - L_0 \right) \frac{1}{\sqrt{d^2 + y^2}} \frac{dy}{dt} \frac{dy}{dt} = 0
\]

\[
m \frac{d^2 y}{dt^2} + k_1 y + 2 k_2 \left( y - \frac{Log \left( \frac{L_0 y}{\sqrt{d^2 + y^2}} \right)}{\sqrt{d^2 + y^2}} \right) = 0
\]

\[ \text{Nonlinear - use Taylor} \]

\[
\frac{Log \left( \frac{L_0 y}{\sqrt{d^2 + y^2}} \right)}{\sqrt{d^2 + y^2}} \approx 0 + \int \left[ \frac{Log \left( \frac{L_0 y}{\sqrt{d^2 + y^2}} \right)}{\sqrt{d^2 + y^2}} - \frac{Log \left( \frac{L_0 y}{\sqrt{d^2 + y^2}} \right)}{\sqrt{d^2 + y^2}} \right] dy \]

\[
= f^\prime(0) \approx \frac{Log(L_0)}{d}
\]
\[ \sum m \frac{d^2y}{dt^2} + k_1 y + 2k_2 \left(1 - \frac{L_0}{d}\right) y = 0 \]

\[ \Rightarrow \frac{m}{W_n^2} \frac{d^2y}{dt^2} + \frac{1}{W_n^2} k_1 + 2k_2 \left(1 - \frac{L_0}{d}\right) y = 0 \]

\[ W_n = \sqrt{\frac{k_1 + 2k_2 \left(1 - \frac{L_0}{d}\right)}{m}} \]

By choosing \( k_1 + 2k_2 \left(1 - \frac{L_0}{d}\right) = 0 \) we get \( W_n = 0 \) gives perfect isolation.
3.7 Damped Vibration

Goal: Understand effects of energy loss on vibration.

Preliminary: The dashpot

Exerts a force proportional to stretch rate

\[ F_d = c \frac{dL}{dt} \]

\( c = "\text{Dashpot coefficient}\) Ns/m

Can combine like springs

\[ \frac{1}{C_{\text{eff}}} = \frac{1}{C_1} + \frac{1}{C_2} \]

\[ C_{\text{eff}} = C_1 + C_2 \]
Power needed to stretch dashpot

\[ P = F \cdot \nu = -c \frac{dk}{dt} \cdot \frac{dk}{dt} \]

\[ \nu = \frac{dk}{dt} \]

\[ = -c \left( \frac{dk}{dt} \right)^2 \]

Always \( < 0 \) \Rightarrow always dissipating energy
**Canonical damped vibration problem:** The spring mass system is released with velocity $v_0$ from position $s_0$ at time $t=0$. Find $s(t)$.

\[ F = ma \]

\[ F_s = k(s - s_0) \]
\[ F_d = c \frac{ds}{dt} \]

\[ m \frac{d^2s}{dt^2} = -F_s - F_d \]

\[ = m \frac{d^2s}{dt^2} = -k(s - s_0) - c \frac{ds}{dt} \]

\[ \frac{m}{k} \frac{d^2s}{dt^2} + \frac{c}{k} \frac{ds}{dt} + s = s_0 \]

**Approach**

1. $F = ma$
2. EOM
3. Find solution in tables
Standard differential equations for vibration problems

(see pdf on web-site for solutions)

CASE I: \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + x = C \]

CASE II: \[ \frac{1}{\alpha^2} \frac{d^2x}{dt^2} - x = C \]

CASE III: \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\xi}{\omega_n} \frac{dx}{dt} + x = C \]

CASE IV: \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\xi}{\omega_n} \frac{dx}{dt} + x = C + KF(t) \text{ with } F(t) = F_0 \sin \omega t \]

CASE V: \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\xi}{\omega_n} \frac{dx}{dt} + x = C + K \left( y + \frac{2\xi}{\omega_n} \frac{dy}{dt} \right) \text{ with } y = Y_0 \sin \omega t \]

CASE VI: \[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\xi}{\omega_n} \frac{dx}{dt} + x = C - \frac{K}{\omega_n^2} \frac{d^2y}{dt^2} \text{ with } y = Y_0 \sin \omega t \]

Our eq: \[ \frac{M}{k} \frac{d^2s}{dt^2} + \frac{C}{k} \frac{ds}{dt} + s = 0 \]

\[ \frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + \frac{2\xi}{\omega_n} \frac{dx}{dt} + x = C \]

\[ S = x \]

\[ L_0 = C \]

\[ \omega_n = \sqrt{\frac{k}{m}} \]

“Undamped” nat. freq.

\[ \frac{2\xi}{\omega_n} = \frac{C}{k} \Rightarrow S = \frac{C}{2k} \omega_n \]

\[ S = \frac{C}{2\sqrt{km}} \]

“Damping coefficient”
Confusing terminology in damped vibrations

The constant $c$:
(Called dashpot coefficient, or damping coefficient)

The coefficient $\zeta$
(Called 'Damping ratio' or 'Damping factor' or 'Damping coefficient')

\[
\zeta = \frac{c}{2\sqrt{km}} \quad \omega_n = \sqrt{\frac{k}{m}}
\]

\[
\frac{1}{\omega_n^2} \frac{d^2x}{dt^2} + 2\zeta \frac{dx}{dt} + x = 0
\]

Some texts also call the product $\zeta \omega_n$ the 'damping coefficient'
Solution to Case III (From handout)

Solution to \[ \frac{1}{\omega_n^2} \frac{d^2 x}{dt^2} + \frac{2 \zeta}{\omega_n} \frac{dx}{dt} + x = C \] with \[ x = x_0, \quad \frac{dx}{dt} = v_0, \quad t = 0 \]

Let \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) (Note absolute value)

Overdamped \( \zeta > 1 \)
\[ x(t) = C + \exp(-\zeta \omega_n t) \left\{ \frac{v_0 + (\zeta \omega_n + \omega_d)(x_0 - C)}{2 \omega_d} \exp(\omega_d t) - \frac{v_0 + (\zeta \omega_n - \omega_d)(x_0 - C)}{2 \omega_d} \exp(-\omega_d t) \right\} \]

Critically Damped \( \zeta = 1 \)
\[ x(t) = C + \left\{ (x_0 - C) + [v_0 + \omega_n (x_0 - C)] t \right\} \exp(-\omega_n t) \]

Underdamped \( \zeta < 1 \)
\[ x(t) = C + \exp(-\zeta \omega_n t) \left\{ (x_0 - C) \cos \omega_d t + \frac{v_0 + \zeta \omega_n (x_0 - C)}{\omega_d} \sin \omega_d t \right\} \]
Underdamped Solution \( \delta < 1 \)

\[ x(t) = A \exp(-\delta \omega_d t) \sin(\omega_d t + \phi) \]

\[ \omega_d = \omega_n \sqrt{1 - \delta^2} \]

Slightly smaller than \( \omega_n \)

Exponentially decaying vibration

Frequency \( \omega_d \) (close to \( \omega_n \))

Decays as \( \exp(-\delta \omega_d t) \)

Faster decay for larger \( \delta \)

Fastest decay for \( \delta = 1 \) — “Critical Damping”
Overdamped case $S > 1$

\[
x(t) = \exp(-SW_n t) \left\{ A \exp(W_n t) + B \exp(-W_n t) \right\}
\]

\[
W_n = W_n \sqrt{S^2 - 1}
\]

\[
x(t) = A \exp\left\{-W_n t (S - \sqrt{S^2 - 1})\right\} + B \exp\left\{-W_n t (S + \sqrt{S^2 - 1})\right\}
\]

Decays slowly

Fast decay

Slowest term decays more quickly as $S$ gets smaller

Fastest decay for $S \to 1$

"Critical Damping"