Rigid Body Dynamics

1. Analyzing motion in systems of links
2. Analyzing motion in gears and pulley systems
3. The rolling wheel
Angular velocity vector:
1. Direction – parallel to rotation axis (RH screw rule)
2. Magnitude – angle (radians) turned per sec
   \[ \omega = \frac{d\theta}{dt} \]
   \[ \mathbf{n} = \omega \mathbf{n} \]

Angular acceleration vector:
\[ \alpha = \frac{d\omega}{dt} \]

For planar motion:
\[ \omega = \frac{d\theta}{dt} \]
\[ \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \]
\[ \omega = \frac{d\theta}{dt} \mathbf{k} \]
\[ \alpha = \frac{d^2\theta}{dt^2} \mathbf{k} \]
Rigid body kinematics formulas

Velocities of two points on a rigid body related by

\[ \mathbf{v}_B - \mathbf{v}_A = \mathbf{\omega} \times (\mathbf{r}_B - \mathbf{r}_A) \]

Accelerations of two points on a rigid body related by

\[ \mathbf{a}_B - \mathbf{a}_A = \mathbf{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \mathbf{\omega} \times \left\{ \mathbf{\omega} \times (\mathbf{r}_B - \mathbf{r}_A) \right\} \]

For 2D problems we can use

\[ \mathbf{v}_B - \mathbf{v}_A = \omega \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A) \]

\[ \mathbf{a}_B - \mathbf{a}_A = \alpha \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A) - \omega^2 (\mathbf{r}_B - \mathbf{r}_A) \]

Constraints at connections

No slip

\[ \mathbf{v}_A = \mathbf{v}_B \]

\[ \mathbf{v}_A \cdot \mathbf{n} = \mathbf{v}_B \cdot \mathbf{n} \]

\[ \mathbf{a}_A = \mathbf{a}_B \]

Slip

\[ \mathbf{v}_A \cdot \mathbf{n} = \mathbf{v}_B \cdot \mathbf{n} \]

\[ \mathbf{a}_A \cdot \mathbf{n} = \mathbf{a}_B \cdot \mathbf{n} \]
Motion in systems of rigid bodies

1. Introduce unknown \( \omega, \alpha \) for each rigid body
2. Find points with known velocities
3. Use rigid body kinematics equations and constraint equations at joints to write down known velocities in terms of unknown \( \omega \)
4. Solve vector equations for unknown \( \omega \)
5. Repeat for accelerations

Problem: how to rotate joints at A, B to move end C with prescribed velocity
Link AB rotates counterclockwise with constant angular speed $5 \text{ rad/s}$.
Find the velocity and acceleration of point E.

Note $\mathbf{v}_A = \mathbf{v}_D = \mathbf{0}$

Rigid bodies connected by pin joint have equal $\mathbf{v}$ @ joint.

Use rigid body formulas.

**Velocities**

\[
\mathbf{v}_B - \mathbf{v}_A = 5 \mathbf{k} \times (1 \mathbf{i}) = 5 \mathbf{j}
\]

\[
\mathbf{v}_C - \mathbf{v}_B = \omega_{BC} \mathbf{k} \times (-2 \mathbf{i} + 2 \mathbf{j}) = -2 \omega_{BC} \mathbf{i} - \omega_{BC} \mathbf{j}
\]

\[
\mathbf{v}_D - \mathbf{v}_C = \omega_{CE} \mathbf{k} \times (-2 - 5 \mathbf{j}) = -2 \cdot 5 \omega_{CE} \mathbf{j}
\]

Add $\mathbf{v}_D - \mathbf{v}_A = \mathbf{0} = 0 \mathbf{i} + \mathbf{0} \mathbf{j} = -2 \omega_{BC} \mathbf{i} + (5 - \omega_{BC} - 2 \cdot 5 \omega_{CE}) \mathbf{j}$

$\Rightarrow \omega_{BC} = 0$ $\omega_{CE} = 2 \text{ rad/s}$
\[ \begin{align*}
\vec{a}_B - \vec{a}_A &= 0 \hat{k} \times (1 \hat{i}) - 5^2 (1 \hat{i}) = -25 \hat{i} \\
\vec{a}_C - \vec{a}_B &= \alpha_{BC} \hat{k} \times (-2 \hat{i} + 2 \hat{j}) - 0^2 (-2 \hat{i} + 2 \hat{j}) = -2 \alpha_{BC} \hat{i} - \alpha_{BC} \hat{j} \\
\vec{a}_D - \vec{a}_C &= \alpha_{CE} \hat{k} \times (-2.5 \hat{i}) - 2^2 (-2.5 \hat{i}) = 10 \hat{i} - 2.5 \alpha_{CE} \hat{j} \\
\text{Add:} \quad \vec{a}_D - \vec{a}_A &= 0 = 0 \hat{i} + 0 \hat{j} = (-2 \alpha_{BC} - 15) \hat{i} - (\alpha_{BC} + 2.5 \alpha_{CE}) \hat{j} \\
\alpha_{BC} &= -1.5 \text{ rad/s}^2 \quad \alpha_{CE} = +3 \text{ rad/s}^2 \\
\text{For E use formula again} \quad \vec{v}_E - \vec{v}_D &= 2 \hat{k} \times (-2.5 \hat{i}) = -5 \hat{j} \text{ m/s} \\
\vec{a}_E - \vec{a}_D &= 3 \hat{k} \times (-2.5 \hat{i}) - 2^2 (-2.5 \hat{i}) \\
&= (10 \hat{i} - 5 \hat{j}) \text{ m/s}^2
\end{align*} \]
Gears, Pulleys & the rolling wheel

Simple Gear Pair:

Two gears have same velocity at C

Gear A \( v_c - 0 = w_A R_A (R_c) = R_w A \hat{f} \)

Gear B \( v_c - 0 = w_B R_B (-R_c) = -R_w B \hat{f} \)

\( w_A R_A = -w_B R_B \)

\[
\frac{w_B}{w_A} = -\frac{R_A}{R_B} = -\frac{N_A}{N_B}
\]

Or in terms of \( N_A, N_B \), spacing between teeth equal

\( 2\pi R_A / N_A = 2\pi R_B / N_B \Rightarrow R_A / R_B = N_A / N_B \)
The radii $R_A, R_B$ of two pulleys in a CVT vary with time. If A rotates at constant angular speed $\omega_A$.

Find a formula for the angular acceleration of B $\alpha_B$

All points on belt have same speed

$$\omega_A R_A = \omega_B R_B \implies \omega_B = \frac{\omega_A R_A}{R_B}$$

To find $\alpha_B$; differentiate wrt time

$$\alpha_B = \frac{d\omega_B}{dt} = \frac{\omega_A}{R_B} \left\{ \frac{1}{R_A} \frac{dR_A}{dt} - \frac{R_A}{R_B} \frac{dR_B}{dt} \right\}$$

$$V = \omega_B R_B$$
Simple epicyclic gear problem

The planet carrier is stationary $\omega_{zPC} = 0$

Find the angular speeds of the planet $\omega_{zP}$ and ring gear $\omega_{zR}$

Here planet gear center stationary $\implies$ Planet & sun are a standard gear pair

Geometry $\implies$ radius of planet $r = (R_R - R_S)/2 \quad N_p = (N_R - N_S)/2$

$$\frac{\omega_{zp}}{\omega_{zS}} = -\frac{R_S}{r}$$

At C planet & ring have same speed $\omega_{zP} \Gamma = R_R \omega_{zR}$
Combine

\[
\frac{W_2R}{W_2s} = -\frac{Rs}{RR} = -\frac{Ns}{NR}
\]

### General epicyclic gear problem

The planet carrier has angular speed \( \omega_{zPC} \)

The sun gear has angular speed \( \omega_{zS} \)

Find the angular speeds of the planet \( \omega_{zP} \)
and ring gear \( \omega_{zR} \)

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**Trick:** Adopt rotating reference frame rotating with planet carrier
⇒ Subtract \( W_{pe} \) from all angular velocities

⇒ Planet carrier stationary

⇒ Use formulas for stationary carrier to relate modified angular speeds

\[
\frac{W_{2p} - W_{2pc}}{W_{2s} - W_{2pc}} = -\frac{R_s}{R}
\]

\[
\frac{W_{2R} - W_{2pc}}{W_{2s} - W_{2pc}} = -\frac{R_s}{R} = \frac{-N_s}{N_R}
\]
Rolling Wheel Formula

Find formulas relating $V_0$, $\alpha_0$ to $\omega_2$, $\alpha_2$

Physics: Point C on wheel is instantaneously stationary $V_C = 0$

Kinematic formula $V_0 - V_C = \omega_2 \vec{r} \times (\vec{R}_i)$

$\implies V_0 = -\omega_2 R \vec{i}$

$V_{0x} = -\omega_2 R$

Accel: Take time derivative

$\alpha_0 = -\alpha_2 R \vec{i}$

$\alpha_{0x} = -\alpha_2 R$

For other points use formula eg $V_A = V_0 + \omega_2 \vec{r} \times (-\vec{R}_i)$

$\implies V_A = -\omega_2 R (\vec{i} + \vec{j})$