Chapter 4
Conservation laws for systems of particles

In this chapter, we shall introduce the following general concepts:

1. The power, or rate of work done by a force
2. The total work done by a force
3. The kinetic energy of a particle
4. The power-kinetic energy and work-kinetic energy relations for a single particle
5. The concepts of a *conservative force* and a *conservative system*
6. The power-total energy and work-total energy relations for a conservative system
7. Energy conservation for a conservative system
8. The linear impulse of a force
9. The linear momentum of a particle (or system of particles)
10. The linear impulse - linear momentum relations for a single particle
11. Linear impulse-momentum relations for a system of particles
12. Conservation of linear momentum for a system
13. Analyzing collisions between particles using linear momentum
14. The angular impulse of a force
15. The angular momentum of a particle
16. The angular impulse – angular momentum relation for a single particle

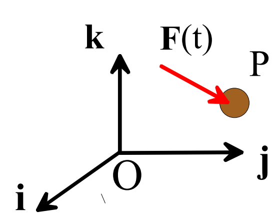
We will also illustrate how these concepts can be used in engineering calculations. As you will see, to applying these principles to engineering calculations you will need two things: (i) a thorough understanding of the principles themselves; and (ii) Physical insight into how engineering systems behave, so you can see how to apply the theory to practice. The first is easy. The second is hard, but practice will help.

4.1 Work, Power, Potential Energy and Kinetic Energy relations for particles

The concepts of work, power and energy are among the most powerful ideas in the physical sciences. Their most important application is in the field of *thermodynamics*, which describes the exchange of energy between interacting systems. In addition, concepts of energy carry over to relativistic systems and quantum mechanics, where the classical versions of Newton's laws themselves no longer apply.

In this section, we develop the basic definitions of mechanical work and energy, and show how they can be used to analyze motion of dynamical systems. Future courses will expand on these concepts further.

4.1.1 Definition of the power and work done by a force



Suppose that a force **F** acts on a particle that moves with speed **v**.

By definition:

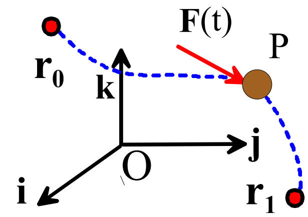
- The *Power* developed by the force, (or the *rate of work done* by the force) is $P = \mathbf{F} \cdot \mathbf{v}$.

  If both force and velocity are expressed in Cartesian components, then

$$P = F_x v_x + F_y v_y + F_z v_z$$

Power has units of Nm/s, or `Watts' in SI units.

- The work done by the force during a time interval $t_0 \le t \le t_1$ is



$$W = \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v} dt$$

The work done by the force can also be calculated by integrating the force vector along the path traveled by the force, as

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$$

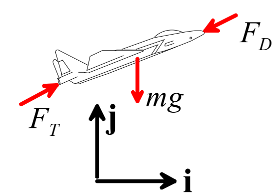
where $\mathbf{r}_0, \mathbf{r}_1$ are the initial and final positions of the force.

Work has units of Nm in SI units, or `Joules'

A moving force can do work on a particle, or on any moving object. For example, if a force acts to stretch a spring, it is said to do work on the spring.

4.1.2 Simple examples of power and work calculations

Example 1: An aircraft with mass 45000 kg flying at 200 knots (102m/s) climbs at 1000ft/min. Calculate the rate of work done on the aircraft by gravity.

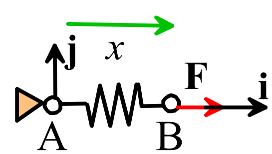


The gravitational force is $-mg\mathbf{j}$, and the velocity vector of the aircraft is $\mathbf{v} = v_x \mathbf{i} + v_y \mathbf{j}$. The rate of work done on the aircraft is therefore

$$P = \mathbf{F} \cdot \mathbf{v} = -mgv_y$$

Substituting numbers gives

Example 2: Calculate a formula for the work required to stretch a spring with stiffness k and unstretched length L_0 from length l_0 to length l_1.



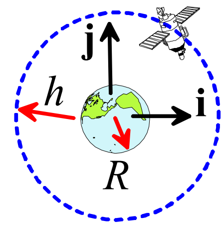
The figure shows a spring that held fixed at A and is stretched in the horizontal direction by a force $\mathbf{F} = F_x \mathbf{i}$ acting at B. At some instant the spring has length $l_0 < x < l_1$. The spring force law states that the force acting on the spring at B is related to the length of the spring x by

$$\mathbf{F} = F_x \mathbf{i} = k(x - L_0)\mathbf{i}$$

The position vector of the force is $\mathbf{r} = x\mathbf{i}$, and therefore the work done is

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{l_0}^{l_1} k(x - L_0)\mathbf{i} \cdot dx\mathbf{i} = \frac{k}{2}\left\{(l_1 - L_0)^2 - (l_0 - L_0)^2\right\}$$

Example 3: Calculate the work done by gravity on a satellite that is launched from the surface of the earth to an altitude of 250km (a typical low earth orbit).



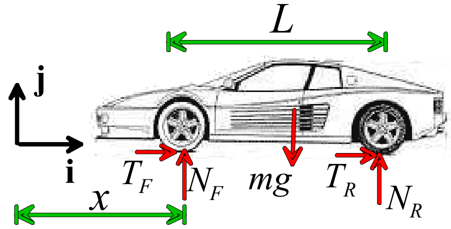
Assumptions

1. The earth's radius is 6378.145km
2. The mass of a typical satellite is 4135kg - see , e.g. http://www.astronautix.com/craft/hs601.htm
3. The Gravitational parameter $\mu = GM = 3.986012 \times 10^5 \ \text{km}^3\text{s}^{-1}$ (G= gravitational constant; M=mass of earth)
4. We will assume that the satellite is launched along a straight line path parallel to the **i** direction, starting the earths surface and extending to the altitude of the orbit. It turns out that the work done is independent of the path, but this is not obvious without more elaborate and sophisticated calculations.

Calculation:

1. The gravitational force on the satellite is
$$\mathbf{F} = -\frac{GMm}{x^2}\mathbf{i}$$
2. The work done follows as
$$\mathbf{F} = -\int_R^{R+h} \frac{GMm}{x^2}\mathbf{i}\cdot dx\mathbf{i} = GMm\left(\frac{1}{R+h} - \frac{1}{R}\right)$$
3. Substituting numbers gives 9.7×10^6 J (be careful with units – if you work with kilometers the work done is in N-km instead of SI units Nm)

Example 4: A Ferrari Testarossa skids to a stop over a distance of 250ft. Calculate the total work done on the car by the friction forces acting on its wheels.



Assumptions:

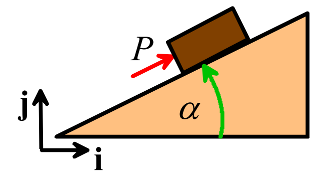
1. A Ferrari Testarossa has mass 1506kg (see http://www.ultimatecarpage.com/car/1889/Ferrari-Testarossa.html)
2. The coefficient of friction between wheels and road is of order 0.8
3. We assume the brakes are locked so all wheels skid, and air resistance is neglected

Calculation The figure shows a free body diagram. The equation of motion for the car is
$$(T_F + T_R)\mathbf{i} + (N_R + N_F - mg)\mathbf{j} = ma_x\mathbf{i}$$

1. The vertical component of the equation of motion yields $N_R + N_F = mg$
2. The friction law shows that $T_R = \mu N_R \quad T_F = \mu N_F \Rightarrow \quad T_R + T_F = \mu\left(N_R + N_F\right) = \mu mg$
3. The position vectors of the car's front and rear wheels are $\mathbf{r}_F = x\mathbf{i} \qquad \mathbf{r}_R = (x+L)\mathbf{i}$. The work done follows as. We suppose that the rear wheel starts at some point $x_0\mathbf{i}$ when the brakes are applied and skids a total distance d.
$$W = \int_{x_0}^{x_0-d} \mathbf{F}_R\cdot d\mathbf{r}_R + \int_{x_0+L}^{x_0-d+L} \mathbf{F}_F\cdot d\mathbf{r}_F = \int_{x_0}^{x_0-d} T_R\mathbf{i}\cdot(dx\mathbf{i}) + \int_{x_0+L}^{x_0-d+L} T_F\mathbf{i}\cdot(dx\mathbf{i}) = -(T_R+T_F)d$$

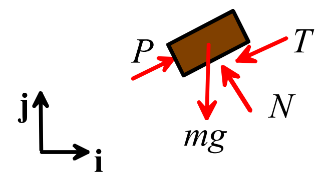
4. The work done follows as $W = -\mu mg$. Substituting numbers gives $W = 9\times 10^5 J$.

Example 5: The figure shows a box that is pushed up a slope by a force *P*. The box moves with speed *v*. Find a formula for the rate of work done by each of the forces acting on the box.



The figure shows a free body diagram. The force vectors are

1. Applied force $P\cos\alpha\mathbf{i} + P\sin\alpha\mathbf{j}$
2. Friction $-T\cos\alpha\mathbf{i} - T\sin\alpha\mathbf{j}$
3. Normal reaction $-N\sin\alpha\mathbf{i} + T\cos\alpha\mathbf{j}$
4. Weight $-mg\mathbf{j}$



The velocity vector is $v\cos\alpha\mathbf{i} + v\sin\alpha\mathbf{j}$

Evaluating the dot products $\mathbf{F}\cdot\mathbf{v}$ for each formula, and recalling that $\cos^2\alpha + \sin^2\alpha = 1$ gives

1. Applied force Pv
2. Friction $-Tv$
3. Normal reaction 0
4. Weight $-mg\sin\alpha$

Example 6: The table lists the experimentally measured force-v-draw data for a long-bow. Calculate the total work done to draw the bow.

Force *N*	Draw (cm)
0	0
40	10
90	20
140	30
180	40
220	50
270	60

In this case we don't have a function that specifies the force as a function of position; instead, we have a table of numerical values. We have to approximate the integral

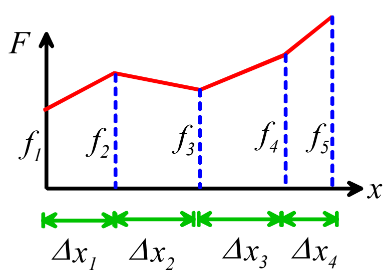
$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{r}$$

numerically. To understand how to do this, remember that integrating a function can be visualized as computing the area under a curve of the function, as illustrated in the figure.

We can estimate the integral by dividing the area into a series of trapezoids, as shown. Recall that the area of a trapezoid is (base x average height), so the total area of the function is

$$W = \Delta x_1\left(\frac{f_1+f_2}{2}\right) + \Delta x_2\left(\frac{f_2+f_3}{2}\right) + \Delta x_3\left(\frac{f_3+f_4}{2}\right) + \Delta x_4\left(\frac{f_4+f_5}{2}\right)$$

$$= 0.1\left(\frac{40+0}{2}\right) + 0.1\left(\frac{20+10}{2}\right) + \ldots.$$

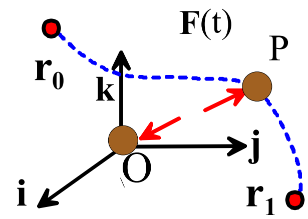


You could easily do this calculation by hand – but for lazy people like me MATLAB has a convenient function called `trapz' that does this calculation automatically. Here's how to use it

```
draw = [0,10,20,30,40,50,60]*0.01;
force = [0,40,90,140,180,220,270];
trapz(draw,force)
ans =  80.5000
```

So the solution is 80.5J

4.1.3 Definition of the potential energy of a conservative force



Preamble: Textbooks nearly always define the `potential energy of a force.' Strictly speaking, we cannot define a potential energy of a single force – instead, we need to define the potential energy of a *pair* of forces. A force can't exist by itself – there must always be an equal and opposite reaction force acting on a second body. In all of the discussion to be presented in this section, ***we implicitly assume that the reaction force is acting on a second body, which is fixed at the origin***. This simplifies calculations, and makes the discussion presented here look like those given in textbooks, but you should remember that the potential energy of a force pair is always a function of the relative positions of the two forces.

With that proviso, consider a force **F** acting on a particle at some position **r** in space. Recall that the work done by a force that moves from position vector $\mathbf{r}_0$ to position vector $\mathbf{r}_1$ is

$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$$

In general, the work done by the force depends on the path between $\mathbf{r}_0$ to $\mathbf{r}_1$. For some special forces, however, the work done is independent of the path. Such forces are said to be *conservative*.

For a force to be conservative:

- The force must be a function only of its position – i.e. it can't depend on the velocity of the force, for example.
- The force vector must satisfy $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$

Examples of conservative forces include gravity, electrostatic forces, and the forces exerted by a spring. Examples of non-conservative (or should that be liberal?) forces include friction, air resistance, and aerodynamic lift forces.

The ***potential energy of a conservative force*** is defined as the negative of the work done by the force in moving from some arbitrary initial position $\mathbf{r}_0$ to a new position $\mathbf{r}$, i.e.

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} + \text{constant}$$

The constant is arbitrary, and the negative sign is introduced by convention (it makes sure that systems try to minimize their potential energy). If there is a point where the force is zero, it is usual to put $\mathbf{r}_0$ at this point, and take the constant to be zero.

Note that

1. The potential energy is a scalar valued function
2. The potential energy is a function only of the position of the force. If we choose to describe position in terms of Cartesian components $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $V(\mathbf{r}) = V(x, y, z)$.
3. The relationship between potential energy and force can also be expressed in *differential* form (which is often more useful for actual calculations) as

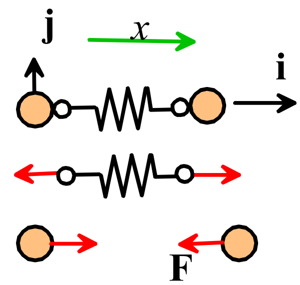
$$\mathbf{F} = -\mathrm{grad}(V)$$

If we choose to work with Cartesian components, then

$$F_x\mathbf{i}+F_y\mathbf{j}+F_z\mathbf{k}=-\left(\frac{\partial V}{\partial x}\mathbf{i}+\frac{\partial V}{\partial y}\mathbf{j}+\frac{\partial V}{\partial z}\mathbf{k}\right)$$

Occasionally, you might have to calculate a potential energy function by integrating forces – for example, if you are interested in running a molecular dynamic simulation of a collection of atoms in a material, you will need to describe the interatomic forces in some convenient way. The interatomic forces can be estimated by doing quantum-mechanical calculations, and the results can be approximated by a suitable potential energy function. Here are a few examples showing how you can integrate forces to calculate potential energy

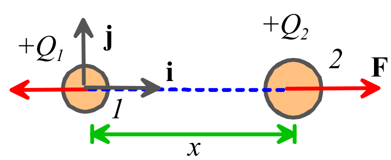
Example 1: Potential energy of forces exerted by a spring. A free body diagram showing the forces exerted by a spring connecting two objects is shown in the figure.



1. The force exerted by a spring is
$$\mathbf{F}=-k(x-L_0)\mathbf{i}$$
2. The position vector of the force is $\mathbf{r}=x\mathbf{i}$
3. The potential energy follows as
$$V(\mathbf{r})=-\int_{\mathbf{r}_0}^{\mathbf{r}}\mathbf{F}\cdot d\mathbf{r}+\text{constant} \;=\int_{L_0}^{L}k(x-L_0)\mathbf{i}\cdot dx\mathbf{i}=\frac{1}{2}k(L-L_0)^2$$
where we have taken the constant to be zero.

Example 2: Potential energy of electrostatic forces exerted by charged particles.

The figure shows two charged particles a distance x apart. To calculate the potential energy of the force acting on particle 2, we place particle 1 at the origin, and note that the force acting on particle 2 is



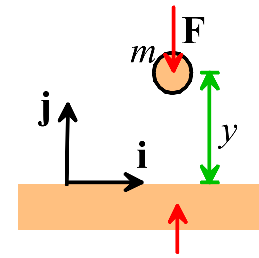
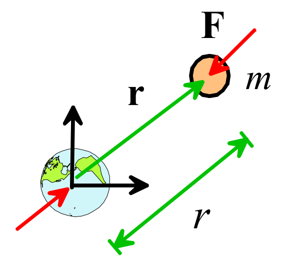
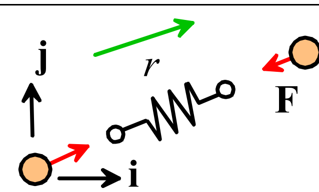
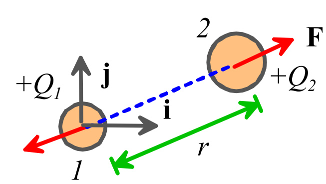
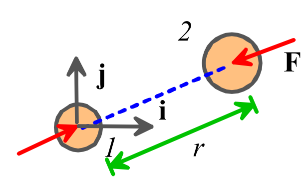
$$\mathbf{F}=-\frac{Q_1Q_2}{4\pi\varepsilon x^2}\mathbf{i}$$

where Q_1 and Q_2 are the charges on the two particles, and ε is a fundamental physical constant known as the *Permittivity* of the medium surrounding the particles. Since the force is zero when the particles are infinitely far apart, we take $\mathbf{r}_0$ at infinity. The potential energy follows as

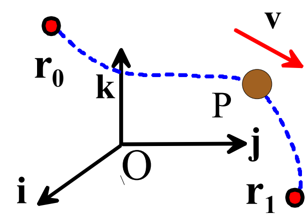
$$V(\mathbf{r})=-\int_{\infty}^{x}\frac{Q_1Q_2}{4\pi\varepsilon x^2}\mathbf{i}\cdot dx\mathbf{i}=\frac{Q_1Q_2}{4\pi\varepsilon x}$$

Table of potential energy relations

In practice, however, we rarely need to do the integrals to calculate the potential energy of a force, because there are very few different kinds of force. For most engineering calculations the potential energy formulas listed in the table below are sufficient.

Type of force	Force vector	Potential energy	
Gravity acting on a particle near earths surface	$\mathbf{F} = -mg\mathbf{j}$	$V = mgy$	
Gravitational force exerted on mass m by mass M at the origin	$\mathbf{F} = -\dfrac{GMm}{r^3}\mathbf{r}$	$V = -\dfrac{GMm}{r}$	
Force exerted by a spring with stiffness k and unstretched length L_0	$\mathbf{F} = -k(r - L_0)\dfrac{\mathbf{r}}{r}$	$V = \dfrac{1}{2}k\left(r - L_0\right)^2$	
Force acting between two charged particles	$\mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon r^3}\mathbf{r}$	$V = \dfrac{Q_1 Q_2}{4\pi\varepsilon r}$	
Force exerted by one molecule of a noble gas (e.g. He, Ar, etc) on another (Lennard Jones potential). a is the equilibrium spacing between molecules, and E is the energy of the bond.	$\mathbf{F} = -12\dfrac{E}{a}\left[2\left(\dfrac{a}{r}\right)^{13} - \left(\dfrac{a}{r}\right)^{7}\right]\dfrac{\mathbf{r}}{r}$	$E\left[\left(\dfrac{a}{r}\right)^{12} - 2\left(\dfrac{a}{r}\right)^{6}\right]$	

4.1.4 Definition of the Kinetic Energy of a particle



Consider a particle with mass m which moves with velocity $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$. By definition, its kinetic energy is

$$T = \frac{1}{2}m|\mathbf{v}|^2 = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right)$$

4.1.5 Power-Work-kinetic energy relations for a single particle

Consider a particle with mass m that moves under the action of a force **F.** Suppose that

1. At some time t_0 the particle has some initial position $\mathbf{r}_0$, velocity $\mathbf{v}_0$ and kinetic energy T_0
2. At some later time t the particle has a new position **r**, velocity $\mathbf{v}$ and kinetic energy T.
3. Let $P = \mathbf{F}\cdot\mathbf{v}$ denote the rate of work done by the force
4. Let $W = \int_{t_0}^{t} P dt = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F}\cdot d\mathbf{r}$ be the total work done by the force

The *Power-kinetic energy* relation for the particle states that the rate of work done by **F** is equal to the rate of change of kinetic energy of the particle, i.e.

$$P = \frac{dT}{dt}$$

Proof: This is just another way of writing Newton's law for the particle: to see this, note that we can take the dot product of both sides of **F**=m**a** with the particle velocity

$$\mathbf{F}\cdot\mathbf{v} = m\mathbf{a}\cdot\mathbf{v} = m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \frac{d}{dt}\left\{\frac{1}{2}m(\mathbf{v}\cdot\mathbf{v})\right\}$$

To see the last step, do the derivative using the Chain rule and note that $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

The *Work-kinetic energy* relation for a particle says that the total work done by the force **F** on the particle is equal to the change in the kinetic energy of the particle.

$$W = T - T_0$$

This follows by integrating the power-kinetic energy relation with respect to time.

4.1.6 Examples of simple calculations using work-power-kinetic energy relations

There are two main applications of the work-power-kinetic energy relations. You can use them to calculate the distance over which a force must act in order to produce a given change in velocity. You can also use them to estimate the energy required to make a particle move in a particular way, or the amount of energy that can be extracted from a collection of moving particles (e.g. using a wind turbine)

Example 1: The longest single-span escalator in the Western hemisphere is located at the Washington Metro station in Montgomery County, Maryland. Some technical specifications for the escalator span can be found on Wikipedia (we have not checked this data!). Additional information concerning escalator standards can be found here.

Calculate the kinetic energy of a single 80kg rider standing on the escalator

The KE is $mv^2/2$; we have that the speed is 27m per minute, so KE is 8.1J.

Calculate the change in potential energy of a single 80kg rider who travels the entire length of the escalator span.

The change in PE is mgh, where $h = 35\text{m}$, so PE is 27468J

Assuming the escalator operates at its theoretical capacity of 9000 passengers per hour, estimate the power required to operate the escalator.

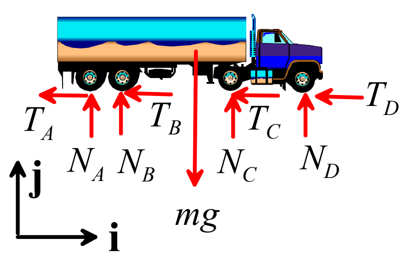
The power is the number of passengers per second multiplied by the energy change per passenger. This gives P=68.7 kW

Example 2: Estimate the minimum distance required for a 14 wheeler that travels at the RI speed-limit to brake to a standstill. Is the distance to stop any different for a Toyota Echo?

This problem can be solved by noting that, since we know the initial and final speed of the vehicle, we can calculate the change in kinetic energy as the vehicle stops. The change in kinetic energy must equal the work done by the forces acting on the vehicle – which depends on the distance slid. Here are the details of the calculation.

Assumptions:

1. We assume that all the wheels are locked and skid over the ground (this will stop the vehicle in the shortest possible distance)
2. The contacts are assumed to have friction coefficient $\mu = 0.8$
3. The vehicle is idealized as a particle.
4. Air resistance will be neglected.



Calculation:

1. The figure shows a free body diagram.
2. The equation of motion for the vehicle is
$$-(T_A + T_B + T_C + T_D)\mathbf{i} + (N_A + N_B + N_C + N_D - mg)\mathbf{j} = ma_x\mathbf{i}$$
The vertical component of the equation shows that $N_A + N_B + N_C + N_D = mg$.
3. The friction force follows as $(T_A + T_B + T_C + T_D) = \mu(N_A + N_B + N_C + N_D) = \mu mg$

4. If the vehicle skids for a distance *d*, the total work done by the forces acting on the vehicle is
$$W = \int_0^d -(T_A + T_B + T_C + T_D)\mathbf{i} \cdot dx\mathbf{i} = -(T_A + T_B + T_C + T_D)d = -\mu mgd$$
5. The work-energy relation states that the total work done on the particle is equal to its change in kinetic energy. When the brakes are applied the vehicle is traveling at the speed limit, with speed *V*; at the end of the skid its speed is zero. The change in kinetic energy is therefore $\Delta T = 0 - mV^2 / 2$. The work-energy relation shows that
$$-mgd = -mV^2 / 2 \Rightarrow d = V^2 / \mu g$$
Substituting numbers gives

This simple calculation suggests that the braking distance for a vehicle depends only on its speed and the friction coefficient between wheels and tires. This is unlikely to vary much from one vehicle to another. In practice there may be more variation between vehicles than this estimate suggests, partly because factors like air resistance and aerodynamic lift forces will influence the results, and also because vehicles usually don't skid during an emergency stop (if they do, the driver loses control) – the nature of the braking system therefore also may change the prediction.

Example 3: Compare the power consumption of a Ford Excursion to that of a Chevy Cobalt during stop-start driving in a traffic jam.

During stop-start driving, the vehicle must be repeatedly accelerated to some (low) velocity; and then braked to a stop. Power is expended to accelerate the vehicle; this power is dissipated as heat in the brakes during braking. To calculate the energy consumption, we must estimate the energy required to accelerate the vehicle to its maximum speed, and estimate the frequency of this event.

Calculation/Assumptions:
1. We assume that the speed in a traffic jam is low enough that air resistance can be neglected.
2. The energy to accelerate to speed *V* is $mV^2 / 2$.
3. We assume that the vehicle accelerates and brakes with constant acceleration – if so, its average speed is *V*/2.
4. If the vehicle travels a distance *d* between stops, the time between two stops is 2*d/V.*
5. The average power is therefore $mV / 4d$.
6. Taking *V=15*mph (7m/s) and *d=200*ft (61m) are reasonable values – the power is therefore 0.03*m*, with *m* in kg. A Ford Excursion weighs 9200 lb (4170 kg), requiring 125 Watts (about that of a light bulb) to keep moving. A Chevy Cobalt weighs 2681lb (1216kg) and requires only 36 Watts – a very substantial energy saving.

Reducing vehicle weight is the most effective way of improving fuel efficiency during slow driving, and also reduces manufacturing costs and material requirements. Another, more costly, approach is to use a system that can recover the energy during braking – this is the main reason that hybrid vehicles like the Prius have better fuel economy than conventional vehicles.

Example 4: Estimate the power that can be generated by a wind turbine.

The figure shows a wind turbine. The turbine blades deflect the air flowing past them: this changes the air speed and so exerts a force on the blades. If the blades move, the force exerted by the air on the blades does work – this work is the power generated by the turbine. The rate of work done by the air on the blades must equal the change in kinetic energy of the air as it flows past the blades. Consequently, we can estimate the power generated by the turbine by calculating the change in kinetic energy of the air flowing through it.

To do this properly needs a very sophisticated analysis of the air flow around the turbine. However, we can get a rather crude estimate of the power by assuming that the turbine is able to extract all the energy from the air that flows through the circular area swept by the blades.

Calculation: Let V denote the wind speed, and let ρ denote the density of the air.

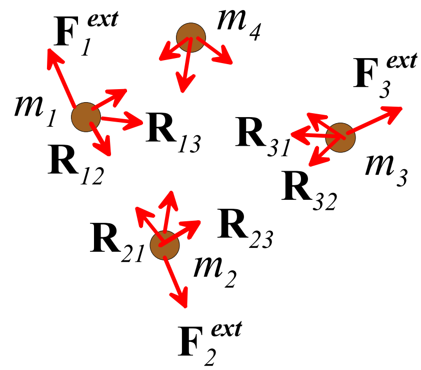
1. In a time t, a cylindrical region of air with radius R and height Vt passes through the fan.
2. The cylindrical region has mass $\pi R^2 \rho V t$
3. The kinetic energy of the cylindrical region of air is $T = \frac{1}{2}\left(\pi R^2 \rho V t\right) V^2$
4. The rate of flow of kinetic energy through the fan is therefore $\frac{dT}{dt} = \frac{\pi}{2} R^2 \rho V^3$
5. If all this energy could be used to do work on the fan blades the power generated would be $P = \frac{dT}{dt} = \frac{\pi}{2} R^2 \rho V^3$

Representative numbers are (i) Air density 1.2 kg/m^3; (ii) air speed 25mph (11 m/sec); (iii) Radius 30m This gives 1.8MW. For comparison, a nuclear power plant generates about 500-1000 MW.

A more sophisticated calculation (which will be covered in EN810) shows that in practice the maximum possible amount of energy that can be extracted from the air is about 60% of this estimate. On average, a typical household uses about a kW of energy; so a single turbine could provide enough power for about 5-10 houses.

4.1.7 Energy relations for a conservative system of particles.

The figure shows a `system of particles' – this is just a collection of objects that we might be interested in, which can be idealized as particles. Each particle in the system can experience forces applied by:



- ***Other particles in the system*** (e.g. due to gravity, electric charges on the particles, or because the particles are physically connected through springs, or because the particles collide). We call these ***internal forces*** acting in the system. We will denote the internal force exerted by the *i*th particle on the *j*th particle by $\mathbf{R}_{ij}$. Note that, because every action has an equal and opposite reaction, the force exerted on the *j*th particle by the *i*th particle must be equal and opposite, to $\mathbf{R}_{ij}$, i.e. $\mathbf{R}_{ij} = -\mathbf{R}_{ji}$.

- ***Forces exerted on the particles by the outside world*** (e.g. by externally applied gravitational or electromagnetic fields, or because the particles are connected to the outside world through mechanical linkages or springs). We call these ***external forces*** acting on the system, and we will denote the external force on the *i* th particle by $\mathbf{F}_i^{ext}(t)$

We define the *rate of external work* (or external power) done on the system as

$$P^{ext} = \sum_{forces} \mathbf{F}_i^{ext}(t) \cdot \mathbf{v}_i(t)$$

We define the *total external work done on the system* during a time interval $t_0 \le t \le t_1$ as the sum of the work done by the external forces.

$$\Delta W^{ext} = \sum_{forces} \int_{t_0}^{t_1} \mathbf{F}_i^{ext}(t) \cdot \mathbf{v}(t) dt$$

The total work done can also include a contribution from external moments acting on the system, but we will worry about this when we analyze rigid body motion..

The system of particles is ***conservative*** if all the ***internal*** forces in the system are conservative. This means that the particles must interact through conservative forces such as gravity, springs, electrostatic forces, and so on. The particles can also be connected by rigid links, or touch one another, but ***contacts between particles must be frictionless***.

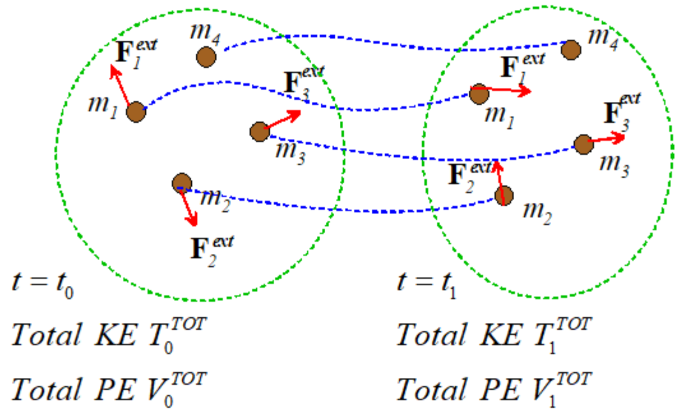
If this is the case, we can define the *total potential energy* of the system V^{TOT} as the sum of potential energies of all the internal forces. We usually compute the total potential energy by summing up all the terms from the table in Sect 4.1.3. Mathematically, the potential energy depends in some complicated way on the distances between all the particles, and the resultant force on the *i*th particle is related to the total potential energy by

$$\sum_{j \ne i} \mathbf{R}_{ij} = -\frac{\partial V^{TOT}}{\partial \mathbf{r}_i}$$

We also define the *total kinetic energy* T^{TOT} of the system as the sum of kinetic energies of all the particles.

$$External\ Power\ \ P^{ext}(t)$$

$$External\ work\ \ \Delta W^{ext} = \int_{t0}^{t1} P(t)dt$$



The work-energy relation for the system of particles can then be stated as follows. Suppose that

1. At some time t_0 the system has and kinetic energy T_0^{TOT}
2. At some later time t_1 the system has kinetic energy T_1^{TOT}.
3. Let V_0^{TOT} denote the potential energy of the force at time t_0
4. Let V_1^{TOT} denote the potential energy of the force at time t
5. Let ΔW^{ext} denote the total work done on the system between $t_0 \le t \le t_1$

Power Energy Relation: This law states that the rate of external work done on the system is equal to the rate of change of total energy of the system

$$P^{ext} = \frac{d}{dt}\left(T_1^{TOT} + V_1^{TOT}\right)$$

Work Energy Relation: This law states that the external work done on the system is equal to the change in total kinetic and potential energy of the system.

$$\Delta W^{ext} = T_1^{TOT} + V_1^{TOT} - \left(T_0^{TOT} + V_0^{TOT}\right)$$

Energy conservation law For the special case where *no* external forces act on the system, the *total energy of the system is constant*

$$\Delta W^{ext} = 0 \quad \Rightarrow T_1^{TOT} + V_1^{TOT} = \left(T_0^{TOT} + V_0^{TOT}\right)$$

It is worth making one final remark before we turn to applications of these laws. We often invoke the principle of conservation of energy when analyzing the motion of an object that is subjected to the earth's gravitational field. For example, the first problem we solve in the next section involves the motion of a projectile launched from the earth's surface. We usually glibly say that `the sum of the potential and kinetic energies of the particle are constant' – and if you've done physics courses you've probably used this kind of thinking. It is not really correct, although it leads to a more or less correct solution.

Properly, we should consider the earth and the projectile together as a conservative system. This means we must include the kinetic energy of the earth in the calculation, which changes by a small, but finite, amount due to gravitational interaction with the projectile. Fortunately, the principle of conservation of linear momentum (to be covered later) can be used to show that the change in kinetic energy of the earth is negligibly small compared to that of the particle.

Proof of the energy relation

Recall the power-kinetic energy relation for a single particle

$$\mathbf{F}\cdot\mathbf{v}=\frac{dT}{dt}$$

The total force on one particle consists of the external force, plus the sum of all the forces exerted on the particle by other particles. The power-energy relation for the i th particle therefore becomes

$$(\mathbf{F}_i^{ext}+\sum_{j\neq i}\mathbf{R}_{ij})\cdot\mathbf{v}_i=\frac{dT_i}{dt}$$

Since the internal forces are conservative, we can write

$$\sum_{j\neq i}\mathbf{R}_{ij}=-\frac{\partial V^{TOT}}{\partial \mathbf{r}_i}$$

Furthermore, $\mathbf{v}_i=\partial\mathbf{r}_i/dt$, so that

$$\sum_{j\neq i}\mathbf{R}_{ij}\cdot\mathbf{v}_i=-\frac{\partial V^{TOT}}{\partial \mathbf{r}_i}\frac{d\mathbf{r}_i}{dt}$$

We can now sum this over all the particles

$$\sum_i \mathbf{F}_i^{ext}\cdot\mathbf{v}_i+\sum_i -\frac{\partial V^{TOT}}{\partial \mathbf{r}_i}\frac{d\mathbf{r}_i}{dt}=\sum_i\frac{dT_i}{dt}$$

We know that V^{TOT} depends only on the positions of the particles, so the second term on the left hand side is a total differential

$$\sum_i -\frac{\partial V^{TOT}}{\partial \mathbf{r}_i}\frac{d\mathbf{r}_i}{dt}=-\frac{dV^{TOT}}{dt}$$

(by the chain rule). We also recognize the first term as the total external power, and the right hand side as the time derivative of the total KE, so we see that

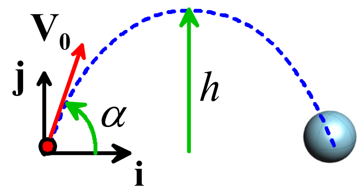
$$P^{ext}-\frac{dV^{TOT}}{dt}=\frac{dT^{TOT}}{dt}$$

Rearranging this gives the power-KE relation for the system, and integrating it with respect to time gives the work-energy relation.

4.1.8 Examples of calculations using kinetic and potential energy in conservative systems

The kinetic-potential energy relations can be used to quickly calculate relationships between the velocity and position of an object. Several examples are provided below.

Example 1: (Boring FE exam question) A projectile with mass m is launched from the ground with velocity V_0 at angle α. Calculate an expression for the maximum height reached by the projectile.



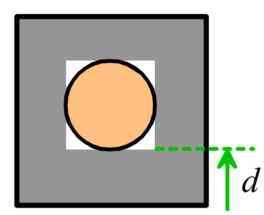
If air resistance can be neglected, we can regard the earth and the projectile together as a conservative system. We neglect the change in the earth's kinetic energy. In addition, since the gravitational force acting on the particle is vertical, the particle's horizontal component of velocity must be constant.

Calculation:

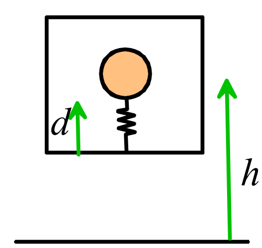
1. Just after launch, the velocity of the particle is $\mathbf{v} = V_0 \cos\alpha \mathbf{i} + V_0 \sin\alpha \mathbf{j}$
2. The kinetic energy of the particle just after launch is $mV_0^2/2$. Its potential energy is zero.
3. At the peak of the trajectory the vertical velocity is zero. Since the horizontal velocity remains constant, the velocity vector at the peak of the trajectory is $\mathbf{v} = V_0 \cos\alpha \mathbf{i}$. The kinetic energy at this point is therefore $mV_0^2 \cos^2\alpha / 2$
4. Energy is conserved, so

$$mV_0^2/2 = mgh + mV_0^2 \cos^2\alpha / 2$$
$$\Rightarrow h = V_0^2(1-\cos^2\alpha)/2 = V_0^2 \sin^2\alpha$$

Example 2: You are asked to design the packaging for a sensitive instrument. The packaging will be made from an elastic foam, which behaves like a spring. The specifications restrict the maximum acceleration of the instrument to 15g. Estimate the thickness of the packaging that you must use.



This problem can be solved by noting that (i) the max acceleration occurs when the packaging (spring) is fully compressed and so exerts the maximum force on the instrument; (ii) The velocity of the instrument must be zero at this instant, (because the height is a minimum, and the velocity is the derivative of the height); and (iii) The system is conservative, and has zero kinetic energy when the package is dropped, and zero kinetic energy when the spring is fully compressed.

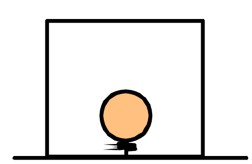


Assumptions:

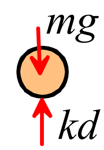
1. The package is dropped from a height of 1.5m
2. The effects of air resistance during the fall are neglected
3. The foam is idealized as a linear spring, which can be fully compressed.

Calculations: Let h denote the drop height; let d denote the foam thickness.

1. The potential energy of the system just before the package is dropped is mgh



2. The potential energy of the system at the instant when the foam is compressed to its maximum extent is $\frac{1}{2}kd^2$
3. The total energy of the system is constant, so
$$\frac{1}{2}kd^2 = mgh$$
4. The figure shows a free body diagram for the instrument at the instant of maximum foam compression. The resultant force acting on the instrument is $(kd - mg)\mathbf{j}$, so its acceleration follows as $\mathbf{a} = (kd/m - g)\mathbf{j}$. The acceleration must not exceed 15g, so
$$kd/m \le 16g$$



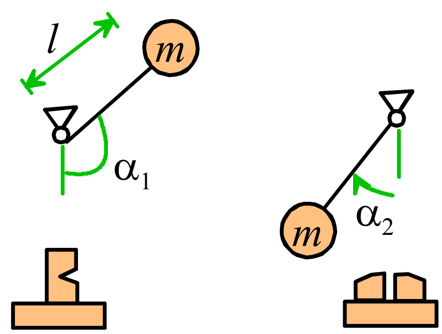
5. Dividing (3) by (4) shows that
$$d \ge h/8$$
The thickness of the protective foam must therefore exceed 18.8cm.

Example 3: The *Charpy Impact Test* is a way to measure the work of fracture of a material (i.e. the work per unit area required to separate a material into two pieces). An example (from **www.qualitest-inc.com/qpi.htm)** is shown in the picture. You can see one in Prince Lab if you are curious.

It consists of a pendulum, which swings down from a prescribed initial angle to strike a specimen. The pendulum fractures the specimen, and then continues to swing to a new, smaller angle on the other side of the vertical. The scale on the pendulum allows the initial and final angles to be measured. The goal of this example is to deduce a relationship between the angles and the work of fracture of the specimen.

The figure shows the pendulum before and after it hits the specimen.

1. The potential energy of the mass before it is released is $V_1 = -mgl\cos\alpha_1$. Its kinetic energy is zero.
2. The potential energy of the mass when it comes to rest after striking the specimen is $V_2 = -mgl\cos\alpha_2$. The kinetic energy is again zero.



The work of fracture is equal to the change in potential energy -
$$W_F = V_1 - V_2 = mgl(\cos\alpha_2 - \cos\alpha_1)$$

Example 4: Estimate the maximum distance that a long-bow can fire an arrow.

We can do this calculation by idealizing the bow as a spring, and estimating the maximum force that a person could apply to draw the bow. The energy stored in the bow can then be estimated, and energy conservation can be used to estimate the resulting velocity of the arrow.

Assumptions

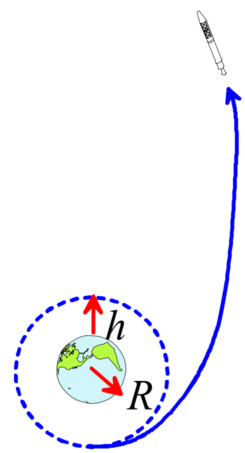
1. The long-bow will be idealized as a linear spring
2. The maximum draw force is likely to be around 60lbf (270N)
3. The draw length is about 2ft (0.6m)
4. Arrows come with various masses – typical range is between 250-600 grains (16-38 grams)
5. We will neglect the mass of the bow (this is not a very realistic assumption)

Calculation: The calculation needs two steps: (i) we start by calculating the velocity of the arrow just after it is fired. This will be done using the energy conservation law; and (ii) we then calculate the distance traveled by the arrow using the projectile trajectory equations derived in the preceding chapter.

1. Just before the arrow is released, the spring is stretched to its maximum length, and the arrow is stationary. The total energy of the system is $T_0 + V_0 = \frac{1}{2}kL^2$, where L is the draw length and k is the stiffness of the bow.
2. We can estimate values for the spring stiffness using the draw force: we have that $F_D = kL$, so $k = F_D / L$. Thus $T_0 + V_0 = \frac{1}{2}LF_D$.
3. Just after the arrow is fired, the spring returns to its un-stretched length, and the arrow has velocity V. The total energy of the system is $T_1 + V_1 = \frac{1}{2}mV^2$, where m is the mass of the arrow
4. The system is conservative, therefore $T_0 + V_0 = T_1 + V_1 \Rightarrow \frac{1}{2}LF_D = \frac{1}{2}mV^2 \Rightarrow V = \sqrt{LF_D / m}$
5. We suppose that the arrow is launched from the origin at an angle θ to the horizontal. The horizontal and vertical components of velocity are $V_x = V\cos\theta \quad V_y = V\sin\theta$. The position vector of the arrow can be calculated using the method outlined in Section 3.2.2 – the result is
$$\mathbf{r} = (Vt\cos\theta)\mathbf{i} + \left(Vt\sin\theta - \frac{1}{2}gt^2\right)\mathbf{j}$$
We can calculate the distance traveled by noting that its position vector when it lands is $d\mathbf{i}$. This gives
$$(Vt\cos\theta)\mathbf{i} + \left(Vt\sin\theta - \frac{1}{2}gt^2\right)\mathbf{j} = d\mathbf{i}$$
where t is the time of flight. The **i** and **j** components of this equation can be solved for t and d, with the result
$$t = \frac{2V\sin\theta}{g} \qquad d = \frac{2V\sin\theta}{g\cos\theta}$$
The arrow travels furthest when fired at an angle that maximizes $\sin\theta / \cos\theta$ - i.e. 45 degrees. The distance follows as
$$d = \frac{2V^2}{g} = \frac{2LF_D}{mg}$$

6. Substituting numbers gives 2064m for a 250 grain arrow – over a mile! Of course air resistance will reduce this value, and in practice the kinetic energy associated with the motion of the bow and bowstring (neglected here) will reduce the distance.

Example 5: Find a formula for the escape velocity of a space vehicle as a function of altitude above the earths surface.



The term 'Escape velocity' means that the space vehicle has a large enough velocity to completely escape the earth's gravitational field – i.e. the space vehicle will never stop after being launched.

Assumptions

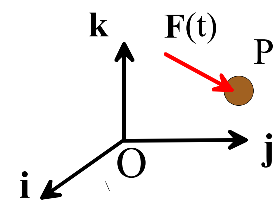
1. The space vehicle is initially in orbit at an altitude h above the earth's surface
2. The earth's radius is 6378.145km
3. While in orbit, a rocket is burned on the vehicle to increase its speed to v (the escape velocity), placing it on a hyperbolic trajectory that will eventually escape the earth's gravitational field.
4. The Gravitational parameter $\mu = GM = 3.986012 \times 10^5 \text{ km}^3\text{s}^{-1}$ (G= gravitational constant; M=mass of earth)
5.

Calculation

1. Just after the rocket is burned, the potential energy of the system is $V = -GMm/(R+h)$, while its kinetic energy is $T = mv^2/2$
2. When it escapes the earth's gravitational field (at an infinite height above the earth's surface) the potential energy is zero. At the critical escape velocity, the velocity of the spacecraft at this point drops to zero. The total energy at escape is therefore zero.
3. This is a conservative system, so
$$T + V = mv^2/2 - GMm/(R+h) = 0$$
$$\Rightarrow v = \sqrt{2GM/(R+h)}$$
4. A typical low earth orbit has altitude of 250km. For this altitude the escape velocity is 10.9km/sec.

4.2 Linear impulse-momentum relations

4.2.1 Definition of the linear impulse of a force



In most dynamic problems, particles are subjected to forces that vary with time. We can write this mathematically by saying that the force is a vector valued function of time $\mathbf{F}(t)$. If we express the force as components in a fixed basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, then

$$\mathbf{F}(t) = F_x(t)\mathbf{i} + F_y(t)\mathbf{j} + F_z(t)\mathbf{k}$$

where each component of the force is a function of time.

The ***Linear Impulse*** exerted by a force during a time interval $t_0 \le t \le t_1$ is defined as

$$\mathfrak{I} = \int_{t_0}^{t_1} \mathbf{F}(t)dt$$

The linear impulse is a vector, and can be expressed as components in a basis

$$\mathfrak{I} = \mathfrak{I}_x\mathbf{i} + \mathfrak{I}_y\mathbf{j} + \mathfrak{I}_z\mathbf{k}$$

$$\mathfrak{I}_x = \int_{t_0}^{t_1} F_x(t)dt \qquad \mathfrak{I}_y = \int_{t_0}^{t_1} F_y(t)dt \qquad \mathfrak{I}_z = \int_{t_0}^{t_1} F_z(t)dt$$

If you know the force as a function of time, you can calculate its impulse using simple calculus. For example:

1. For a *constant* force, with vector value $\mathbf{F}_0$, the impulse is $\mathfrak{I} = \mathbf{F}_0\Delta t$
2. For a *harmonic* force of the form $\mathbf{F}(t) = \mathbf{F}_0 \sin \omega t$ the impulse is

$$\mathfrak{I} = -\mathbf{F}_0\left[\cos(\omega(t_0 + \Delta t)) - \cos \omega t_0\right] / \omega$$

It is rather rare in practice to have to calculate the impulse of a force from its time variation.

4.2.2 Definition of the linear momentum of a particle

The linear momentum of a particle is simply the product of its mass and velocity

$$\mathbf{p} = m\mathbf{v}$$

The linear momentum is a vector – its direction is parallel to the velocity of the particle.

4.2.3 Impulse-momentum relations for a single particle

- Consider a particle that is subjected to a force $\mathbf{F}(t)$ for a time interval $t_0 \le t \le t_1$.
- Let $\mathfrak{I}$ denote the impulse exerted by **F** on the particle
- Let $\mathbf{p}_0$ denote the linear momentum of the particle at time t_0 .
- Let $\mathbf{p}_1$ denote the linear momentum of the particle at time t_1 .

The impulse-momentum equation can be expressed either in differential or integral form.

1. In differential form

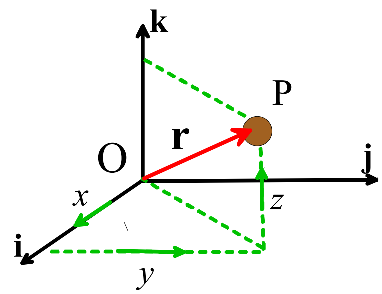
$$\mathbf{F}(t) = \frac{d\mathbf{p}}{dt}$$

This is clearly just a different way of writing Newton's law **F**=m**a**.

2. In integral form

$$\Im = \Delta\mathbf{p}$$

This is the integral of Newton's law of motion with respect to time.



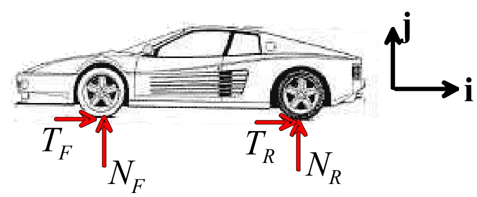
The impulse-momentum relations for a single particle are useful if you need to calculate the change in velocity of an object that is subjected to a prescribed force.

4.2.4 Examples using impulse-momentum relations for a single particle

Example 1: Estimate the time that it takes for a Ferrari Testarossa traveling at the RI speed limit to make an emergency stop. (Like many textbook problems this one is totally unrealistic – nobody in a Ferrari is going to travel at the speed limit!)

We can do this calculation using the impulse-momentum relation for a single particle. Assume that the car has mass m, and travels with speed V before the brakes are applied. Let Δt denote the time required to stop.

1. Start by estimating the force acting on the car during an emergency stop. The figure shows a free body diagram for the car as it brakes to a standstill. We assume that the driver brakes hard enough to lock the wheels, so that the car skids over the road. The horizontal friction forces must oppose the sliding, as shown in the picture. **F**=m**a** for the car follows as



$$\left(T_R + T_F\right)\mathbf{i} + (N_F + N_R - mg)\mathbf{j} = ma_x\mathbf{i}$$

The vertical component of the equation of motion shows that $N_F + N_R = mg$. The friction law then shows that $T_F + T_R = \mu\left(N_F + N_R\right) = \mu mg$

2. The force acting on the car is constant, so the impulse that brings the car to a halt is

$$\Im = \left(T_R + T_F\right)\Delta t\mathbf{i} + (N_F + N_R - mg)\Delta t\mathbf{j} = \mu mg\mathbf{i}$$

3. The linear momentum of the car before the brakes are applied is $\mathbf{p}_0 = -mV\mathbf{i}$. The linear momentum after the car stops is zero. Therefore, $\Delta\mathbf{p} = mV\mathbf{i}$.

4. The linear impulse-momentum relation shows that

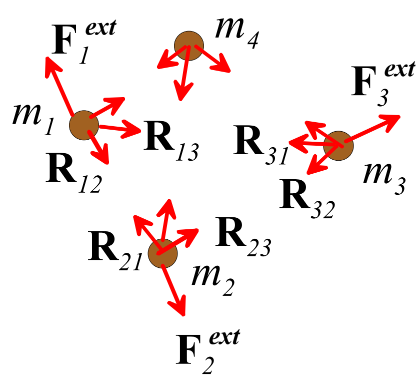
$$\Im = \Delta\mathbf{p} \Rightarrow \mu mg\Delta t\mathbf{i} = mV\mathbf{i}$$

$$\Rightarrow \Delta t = V / (\mu g)$$

5. We can take the friction coefficient as $\mu \approx 0.8$, and 65mph is 29m/s. We take the gravitational acceleration $g = 9.81$. The time follows as $\Delta t \approx 3.7\text{s}$. Note that a TestaRossa can't stop any faster than a Honda Civic, despite the price difference…

4.2.5 Impulse-momentum relation for a system of particles

Suppose we are interested in calculating the motion of several particles, sketched in the figure.



Total external impulse on a system of particles: Each particle in the system can experience forces applied by:

- ***Other particles in the system*** (e.g. due to gravity, electric charges on the particles, or because the particles are physically connected through springs, or because the particles collide). We call these ***internal forces*** acting in the system. We will denote the internal force exerted by the *i*th particle on the *j*th particle by $\mathbf{R}_{ij}$. Note that, because every action has an equal and opposite reaction, the force exerted on the *j*th particle by the *i*th particle must be equal and opposite, to $\mathbf{R}_{ij}$, i.e. $\mathbf{R}_{ij} = -\mathbf{R}_{ji}$.

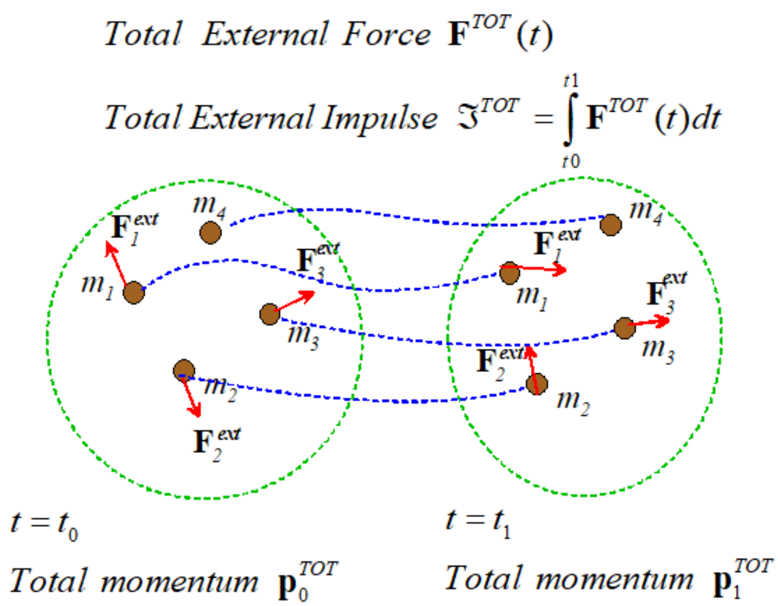
- ***Forces exerted on the particles by the outside world*** (e.g. by externally applied gravitational or electromagnetic fields, or because the particles are connected to the outside world through mechanical linkages or springs). We call these ***external forces*** acting on the system, and we will denote the external force on the *i* th particle by $\mathbf{F}_i^{ext}(t)$

We define the *total impulse* exerted on the system during a time interval $t_0 \le t \le t_1$ as the sum of all the impulses on all the particles. It's easy to see that the total impulse due to the internal forces is zero – because the *i*th and *j*th particles must exert equal and opposite impulses on one another, and when you add them up they cancel out. So the total impulse on the system is simply

$$\Im^{TOT} = \sum_{particles} \int_{t_0}^{t_1} \mathbf{F}_i^{ext}(t)dt$$

Total linear momentum of a system of particles: The total linear momentum of a system of particles is simply the sum of the momenta of all the particles, i.e.

$$\mathbf{p}^{TOT} = \sum_{particles} m_i \mathbf{v}_i$$



The **impulse-momentum equation**

1. In differential form

$$\sum_{particles} \mathbf{F}_i^{ext}(t) = \frac{d\mathbf{p}^{TOT}}{dt}$$

2. In integral form

$$\Im^{TOT} = \mathbf{p}_1^{TOT} - \mathbf{p}_0^{TOT}$$

This is the integral of Newton's law of motion with respect to time.

Conservation of momentum: If *no* external forces act on a system of particles, *their total linear momentum is conserved,* i.e.

$$\mathbf{p}_1^{TOT} = \mathbf{p}_0^{TOT}$$

Deriving the impulse-momentum equation

We start with the impulse-momentum relation for a single particle in differential form

$$\mathbf{F}_i^{ext} + \sum_{j \neq i} \mathbf{R}_{ij} = \frac{d\mathbf{p}_i}{dt}$$

(The left hand side is the total force on the *ith* particle – it includes the external force, as well as all the forces exerted by all the other particles).

Now sum over all the particles

$$\sum_i \left(\mathbf{F}_i^{ext} + \sum_{j \neq i} \mathbf{R}_{ij} \right) = \sum_i \frac{d\mathbf{p}_i}{dt}$$

Notice that since $\mathbf{R}_{ij} = -\mathbf{R}_{ji}$

$$\sum_i \sum_{j \neq i} \mathbf{R}_{ij} = \mathbf{0}$$

(To see this just write out the full sum – every internal force on one particle is equal and opposite to the force on another, so the total has to cancel)

Therefore, evaluating the sums

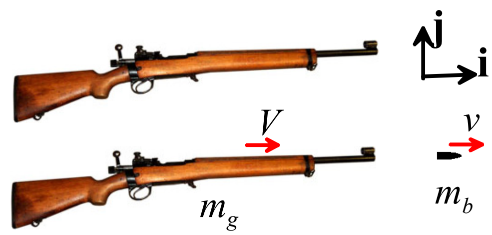
$$\sum_{particles} \mathbf{F}_i^{ext}(t) = \frac{d\mathbf{p}^{TOT}}{dt}$$

We can integrate this expression with respect to time to get the integral version of the theorem.

4.2.6 Examples of applications of momentum conservation for systems of particles

The impulse-momentum equations for systems of particles are particularly useful for (i) analyzing the recoil of a gun; and (ii) analyzing rocket and jet propulsion systems. In both these applications, the internal forces acting between the gun on the projectile, or the motor and propellant, are much larger than any external forces, so the total momentum of the system is conserved.

Example 1: Estimate the recoil velocity of a rifle (youtube abounds with recoil demonstrations – see. e.g. http://www.youtube.com/watch?v=F4juEIK_zRM for samples. Be warned, however – a lot of the videos are tasteless and/or sexist…)



The recoil velocity can be estimated by noting that the total momentum of bullet and rifle together must be conserved. If we can estimate the mass of rifle and bullet, and the bullet's velocity, the recoil velocity can be computed from the momentum conservation equation.

Assumptions:

1. The mass of a typical 0.22 (i.e. 0.22" diameter) caliber rifle bullet is about 7×10^{-4} kg (idealizing the bullet as a sphere, with density 7860 kg/m^3)
2. The muzzle velocity of a 0.22 is about 1000 ft/sec (305 m/s)
3. A typical rifle weighs between 5 and 10 lb (2.5-5 kg)

Calculation:

1. Let m_b denote the bullet mass, and let m_R denote the mass of the rifle.
2. The rifle and bullet are idealized as two particles. Before firing, both are at rest. After firing, the bullet has velocity $\mathbf{v}_b = v\mathbf{i}$; the rifle has velocity $\mathbf{v}_R = V\mathbf{i}$.
3. External forces acting on the system can be neglected, so
$$\mathbf{p}_{tot}(t_0 + \Delta t) = \mathbf{p}_{tot}(t_0) \Rightarrow m_b \mathbf{v}_b + m_R \mathbf{v}_R = \mathbf{0}$$
$$\Rightarrow m_b v\mathbf{i} + m_R V\mathbf{i} = \mathbf{0} \Rightarrow V = -m_b v / m_R .$$
4. Substituting numbers gives $|V|$ between 0.04 and 0.08 m/s (about 0.14 ft/sec)

Example 2: Derive a formula that can be used to estimate the mass of a handgun required to keep its recoil within acceptable limits.

The preceding example shows that the firearm will recoil with a velocity that depends on the ratio of the mass of the bullet to the firearm. The firearm must be brought to rest by the person holding it.

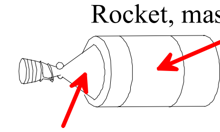
Assumptions:

1. We will idealize a person's hand holding the gun as a spring, with stiffness k, fixed at one end. The 'end-point stiffness' of a human hand has been extensively studied – see, e.g. Shadmehr *et al* J. Neuroscience, **13** (1) 45 (1993). Typical values of stiffness during quasi-static deflections are of order 0.2 N/mm. During dynamic loading stiffnesses are likely to be larger than this.
2. We idealize the handgun and bullet as particles, with mass m_b and m_g, respectively.

Calculation.

1. The preceding problem shows that the firearm will recoil with velocity $V = -m_b v / m_g$
2. Energy conservation can be used to calculate the recoil distance. We consider the firearm and the hand holding it a system. At time $t=0$ it has zero potential energy; and has kinetic energy $T = \frac{1}{2} m_g V^2 = \frac{1}{2}(m_b v)^2 / m_g$. At the end of the recoil, the gun is at rest, and the spring is fully compressed – the kinetic energy is zero, and potential energy is $\frac{1}{2} kd^2$. Energy conservation gives $kd^2 = (m_b v)^2 / m_g$
3. The required mass follows as $m_g = (m_b v)^2 / kd^2$.

Example 4 Rocket propulsion equations. Rocket motors and jet engines exploit the momentum conservation law in order to produce motion. They work by expelling mass from a vehicle at very high velocity, in a direction opposite to the motion of the vehicle. The momentum of the expelled mass must be balanced by an equal and opposite change in the momentum of the vehicle; so the velocity of the vehicle increases.

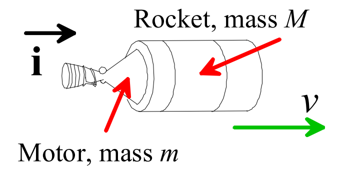


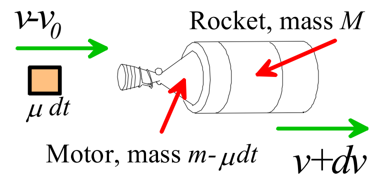
Analyzing a rocket engine is quite complicated, because the propellant carried by the engine is usually a very significant fraction of the total mass of the vehicle. Consequently, it is important to account for the fact that the mass decreases as the propellant is used.

Assumptions:

1. The figure shows a rocket motor attached to a rocket with mass M.
2. The rocket motor contains an initial mass m_0 of propellant and expels propellant at rate $\frac{dm}{dt} = \dot{\mu}_0$ (kg/sec) with a velocity $\mathbf{v}_0 = -v_0 \mathbf{i}$ relative to the rocket.
3. We assume straight line motion, and assume that no external forces act on the rocket or motor.

Calculations:





The figure shows the rocket at an instant of time *t,* and then a very short time interval *dt* later.

1. At time *t*, the rocket moves at speed *v*, and the system has momentum $\mathbf{p} = (M+m)v\mathbf{i}$, where *m* is the motor's mass.
2. During the time interval *dt* a mass $dm = \dot{\mu} dt$ is expelled from the motor. The velocity of the expelled mass is $\mathbf{v} = (v - v_0)\mathbf{i}$.
3. At time *t+dt* the mass of the motor has decreased to $m - \dot{\mu} dt$.
4. At time *t+dt*, the rocket has velocity $\mathbf{v} = (v + dv)\mathbf{i}$.
5. The total momentum of the system at time *t+dt* is therefore
$$\mathbf{p} = (M + m - \mu_0 dt)(v + dv)\mathbf{i} + \mu_0 dt(v - v_0)\mathbf{i}$$
6. Momentum must be conserved, so
$$(M+m)v\mathbf{i} = (M + m - \mu_0 dt)(v + dv)\mathbf{i} + \mu_0 dt(v - v_0)\mathbf{i}$$
7. Multiplying this out and simplifying shows that
$$(M+m)dv\mathbf{i} - \dot{\mu} dt v_0 \mathbf{i} = \mathbf{0}$$
where the term $\mu_0 dtdv$ has been neglected.
8. Finally, we see that
$$(M+m)\frac{dv}{dt} = \dot{\mu} v_0$$

This result is called the `rocket equation.'

***Specific Impulse of a rocket motor*:** The performance of a rocket engine is usually specified by its `specific impulse.' Confusingly, two different definitions of specific impulse are commonly used:

$$I_{sp} = v_0$$

$$\bar{I}_{sp} = \frac{v_0}{g}$$

The first definition quantifies the impulse exerted by the motor per unit mass of propellant; the second is the impulse per unit weight of propellant. You can usually tell which definition is being used from the units – the first definition has units of m/s; the second has units of s. In terms of the specific impulse, the rocket equation is

$$(M+m)\frac{dv}{dt} = \dot{\mu} I_{sp} = \dot{\mu} g \bar{I}_{sp}$$

***Integrated form of the rocket equation*:** If the motor expels propellant at constant rate, the equation of motion can be integrated. Assume that

1. The rocket is at the origin at time *t=0*;
2. The rocket has speed v_0 at time *t=0*
3. The motor has mass m_0 at time *t=0*; this means that at time *t* it has mass $m_0 - \mu t$

Then the rocket's speed can be calculated as a function of time:

$$(M+m_0-\mu t)\frac{dv}{dt}=\dot{\mu}I_{sp} \quad \Rightarrow \int_{v0}^{v} dv=\int_0^t \frac{\dot{\mu}I_{sp}dt}{(M+m_0-\mu t)}$$

$$\Rightarrow v-v_0=I_{sp}\log\left(\frac{M+m_0}{M+m_0-\mu t}\right)$$

Similarly, the position follows as

$$\Rightarrow v=\frac{dx}{dt}=I_{sp}\log\left(\frac{M+m_0}{M+m_0-\mu t}\right)+v_0 \Rightarrow \int_0^x dx=\int_0^t\left(I_{sp}\log\left(\frac{M+m_0}{M+m_0-\mu t}\right)+v_0\right)dt$$

$$\Rightarrow x=(v_0+I_{sp})t-\left(M+m_0-\mu t\right)\frac{I_{sp}}{\mu}\log\left(\frac{M+m_0}{M+m_0-\mu t}\right)$$

These calculations assume that no external forces act on the rocket. It is quite straightforward to generalize them to account for external forces as well – the details are left as an exercise.

Example 5 Application of rocket propulsion equation: Calculate the maximum payload that can be launched to escape velocity on the Ares I launch vehicle.

'Escape velocity' means that after the motor burns out, the space vehicle can escape the earth's gravitational field – see example 5 in Section 4.1.6.

Assumptions

1. The specifications for the Ares I are at http://www.braeunig.us/space/specs/ares.htm Relevant variables are listed in the table below.

	Total mass (kg)	Specific impulse (s)	Propellant mass (kg)
Stage I	586344	268.8	504516
Stage II	183952	452.1	163530

2. As an approximation, we will neglect the motion of the rocket during the burn, and will neglect aerodynamic forces.
3. We will assume that the first stage is jettisoned before burning the second stage.
4. Note that the change in velocity due to burning a stage can be expressed as
$$v-v_0=\overline{I}_{sp}g\log\left(\frac{M_0}{M_0-\Delta m}\right)$$
where M_0 is the total mass before the burn, and Δm is the mass of propellant burned.
5. The earth's radius is 6378.145km
6. The Gravitational parameter $\mu=GM=3.986012\ \times 10^5\ \text{km}^3\text{s}^{-1}$ (G= gravitational constant; M=mass of earth)
7. Escape velocity is from the earths surface is $v_e=\sqrt{2GM/R}$, where R is the earth's radius.

Calculation:

1. Let m denote the payload mass; let m_1, m_2 denote the total masses of stages I and II, let m_{01}, m_{02} denote the propellant masses of stages I and II; and let $\bar{I}_{sp1}$, $\bar{I}_{sp2}$ denote the specific impulses of the two stages.
2. The rocket is at rest before burning the first stage; and its total mass is $m + m_1 + m_2 + m_{01} + m_{02}$. After burn, the mass is $m + m_1 + m_2 + m_{02}$. The velocity after burning the first stage is therefore
$$v_1 = \bar{I}_{sp1} g \log\left(\frac{m + m_1 + m_2 + m_{01} + m_{02}}{m + m_1 + m_2 + m_{02}}\right)$$
3. The first stage is then jettisoned – the mass before starting the second burn is $m + m_2 + m_{02}$, and after the second burn it is $m + m_2$. The velocity after the second burn is therefore
$$v_2 = \bar{I}_{sp1} g \log\left(\frac{m + m_1 + m_2 + m_{01} + m_{02}}{m + m_1 + m_2 + m_{02}}\right) + \bar{I}_{sp2} g \log\left(\frac{m + m_2 + m_{02}}{m + m_2}\right)$$
4. Substituting numbers into the escape velocity formula gives $v_e = 11.18$ km/sec. Substituting numbers for the masses shows that to reach this velocity, the payload mass must satisfy
$$11.18 < 2.621 \log\left(\frac{m + 770296}{m + 265780}\right) + 4.435 \log\left(\frac{m + 183952}{m + 20422}\right)$$
where m is in kg.
5. We can use Matlab to solve for the critical value of m for equality

```
clear all
syms m
eq = 11.1798==2.62908*log((m+770296)/(m+265780))+4.435101*log((m+183952)/(m+20422))
solve(eq,m)
```

```
ans = 8277.5725982589918063415097604449
```

so the solution is 8300kg – a very small mass compared with that of the launch vehicle, but you could pack in a large number of people you would like to launch into outer space nonetheless (the entire faculty of the school of engineering, if you wish).

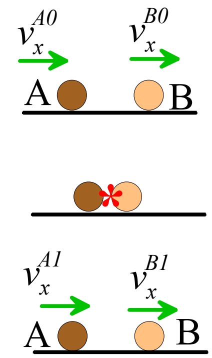
4.2.7 Analyzing collisions between particles: the restitution coefficient

The momentum conservation equations are particularly helpful if you want to analyze collisions between two or more objects. If the impact occurs over a very short time, the impulse exerted by the contact forces acting at the point of collision is huge compared with the impulse exerted by any other forces. If we consider the two colliding particles as a system, the external impulse exerted on the system can be taken to be zero, and so the total momentum of the system is conserved.

The momentum conservation equation can be used to relate the velocities of the particles before collision to those after collision. These relations are not enough to be able to determine the velocities completely, however – to do this, we also need to be able to quantify the energy lost (or more accurately, dissipated as heat) during the collision.

In practice we don't directly specify the energy loss during a collision – instead, the relative velocities are related by a property of the impact called the *coefficient of restitution*.

Restitution coefficient for straight line motion



Suppose that two colliding particles A and B move in a straight line parallel to a unit vector **i.** Let

$\mathbf{v}^{A0} = v_x^{A0}\mathbf{i}$, $\mathbf{v}^{B0} = v_x^{B0}\mathbf{i}$ denote the velocities of A and B just before the collision

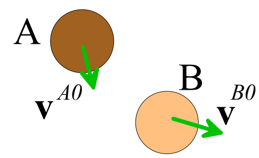
$\mathbf{v}^{A1} = v_x^{A1}\mathbf{i}$, $\mathbf{v}^{B1} = v_x^{B1}\mathbf{i}$ denote the velocities of A and B just after the collision.

The velocities before and after impact are related by

$$v^{B1} - v^{A1} = -e\left(v^{B0} - v^{A0}\right)$$

where e is the restitution coefficient. The negative sign is needed because the particles approach one another before impact, and separate afterwards.

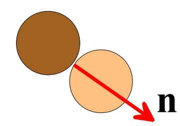
Restitution coefficient for 3D frictionless collisions



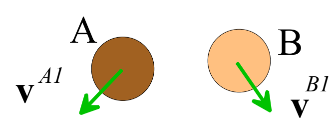
For a more general contact, we define

$\mathbf{v}^{A0}$ $\mathbf{v}^{B0}$ to denote velocities of the particles before collision

$\mathbf{v}^{A1}$ $\mathbf{v}^{B1}$ to denote velocities of the particles after collision



In addition, we let **n** be a unit vector normal to the common tangent plane at the point of contact (if the two colliding particles are spheres or disks the vector is parallel to the line joining their centers).



The velocities before and after impact are related by two vector equations:

$$\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right)\cdot\mathbf{n} = -e\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}$$

$$\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right) - \left[\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right)\cdot\mathbf{n}\right]\mathbf{n} = \left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right) - \left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

To interpret these equations, note that

1. The first equation states that the component of relative velocity normal to the contact plane is reduced by a factor e (just as for 1D contacts)
2. The second equation states that the component of relative velocity tangent to the contact plane is unchanged

To understand the second equation, note that.

- During the collision, a very large force acts on both A and B at the contact between them. Because the contact is frictionless, the direction of the force must be parallel to **n**. Also, the force on A must be equal and opposite to the force on B.
- There is no force acting on either A or B parallel to **t**. This means that momentum must be conserved in the **t** direction for both A and B individually. We can write this mathematically as

$$\mathbf{v}^{B1} - \left[\mathbf{v}^{B1}\cdot\mathbf{n}\right]\mathbf{n} = \mathbf{v}^{B0} - \left[\mathbf{v}^{B0}\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{A1} - \left[\mathbf{v}^{A1}\cdot\mathbf{n}\right]\mathbf{n} = \mathbf{v}^{A0} - \left[\mathbf{v}^{A0}\cdot\mathbf{n}\right]\mathbf{n}$$

(this looks funny, but we have just subtracted the normal component of velocity from the total velocity. This leaves just the tangential component). Subtracting the second equation from the first shows that

$$\left(\mathbf{v}^{B1}-\mathbf{v}^{A1}\right)-\left[\left(\mathbf{v}^{B1}-\mathbf{v}^{A1}\right)\cdot\mathbf{n}\right]\mathbf{n}=\left(\mathbf{v}^{B0}-\mathbf{v}^{A0}\right)-\left[\left(\mathbf{v}^{B0}-\mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

Combined 3D restitution formula

The two equations for the normal and tangential behavior can be combined (just add them) into a single vector equation relating velocities before and after impact – this form is more compact, and often more useful, but more difficult to visualize physically

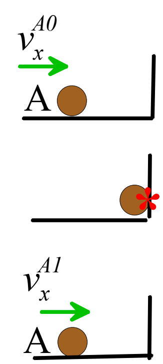
$$\left(\mathbf{v}^{B1}-\mathbf{v}^{A1}\right)=\left(\mathbf{v}^{B0}-\mathbf{v}^{A0}\right)-(1+e)\left[\left(\mathbf{v}^{B0}-\mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

Values of restitution coefficient

The restitution coefficient almost always lies in the range $0\le e\le 1$. It can only be less than zero if one object can penetrate and pass through the other (e.g. a bullet); and can only be greater than 1 if the collision generates energy somehow (e.g. releasing a preloaded spring, or causing an explosion).

If *e=0*, the two colliding objects stick together; if *e=1* the collision is perfectly elastic, with no energy loss.

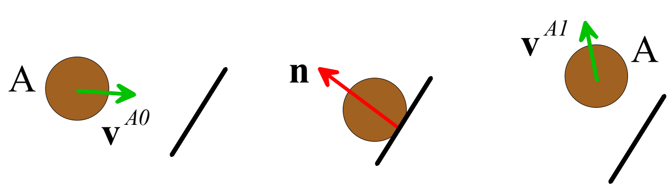
The restitution coefficient is strongly sensitive to the material properties of the two colliding objects, and also varies weakly with their geometry and the velocity of impact. The two latter effects are usually ignored.



Collision between a particle and a fixed rigid surface. The collision formulas can be applied to impact between a rigid fixed surface by taking the surface to be particle *B*, and noting that the velocity of particle *B* is then zero both before and after impact.

For straight line motion, $v^{A1}=-ev^{A0}$

For collision with an angled wall $\mathbf{v}^{A1}=\mathbf{v}^{A0}-(1+e)\left[\mathbf{v}^{A0}\cdot\mathbf{n}\right]\mathbf{n}$, where $\mathbf{n}$ is a unit vector perpendicular to the wall.



4.2.8 Examples of collision problems

Example 1 Suppose that a moving car hits a stationary (parked) vehicle from behind. Derive a formula that will enable an accident investigator to determine the velocity of the moving car from the length of the skid marks left on the road.

Assumptions:

1. We will assume both cars move in a straight line
2. The moving and stationary cars will be assumed to have masses m_1, m_2, respectively
3. We will assume the cars stick together after the collision (i.e. the restitution coefficient is zero)
4. We will assume that only the parked car has brakes applied after the collision

This calculation takes two steps: first, we will use work-energy to relate the distance slid by the cars after impact to their velocity just after the impact occurs. Then, we will use momentum and the restitution formula to work out the velocity of the moving car just before impact.

Calculation: Let V denote the velocity of the moving car just before impact; let v_1 denote the velocity of the two (connected) cars just after impact, and let d denote the distance slid.

1. The figure shows a free body diagram for each of the two cars during sliding after the collision. Newton's law of motion for each car shows that

$$-R\mathbf{i} + \left(N_1 + N_2 - m_1 g\right)\mathbf{j} = m_1 a_x \mathbf{i}$$

$$\left(R + T_3 + T_4\right)\mathbf{i} + \left(N_3 + N_4 - m_2 g\right)\mathbf{j} = m_2 a_x \mathbf{i}$$

2. The vertical component of the equations of motion give $N_1 + N_2 = m_1 g$; $N_3 + N_4 - m_2 g$.
3. The parked car has locked wheels and skids over the road; the friction law gives the tangential forces at the contacts as $T_3 + T_4 = \mu(N_3 + N_4) = \mu m_2 g$
4. We can calculate the velocity of the cars just after impact using the work-kinetic energy relation during skidding. For this purpose, we consider the two connected cars as a single particle. The work done on the particle by the friction forces is $-\left(T_3 + T_4\right)d = -\mu m_2 g d$. The work done is equal to the change in kinetic energy of the particle, so

$$-\mu m_2 g d = 0 - \frac{1}{2}(m_1 + m_2)v_1^2$$

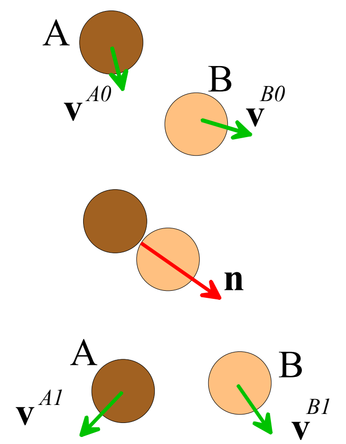
$$v_1 = \sqrt{\frac{2\mu m_2 g d}{m_1 + m_2}}$$

5. Finally, we can use momentum conservation to calculate the velocity just before impact. The momentum after impact is $\mathbf{h}_1 = (m_1 + m_2)v_1\mathbf{i}$, while before impact $\mathbf{h}_0 = m_1 V\mathbf{i}$. Equating the two gives

$$V = \frac{(m_1 + m_2)}{m_1} v_1 = \sqrt{\frac{2\mu(m_1 + m_2)m_2 g d}{m_1^2}}$$

Example 2: Two frictionless spheres with radius R have initial velocity $\mathbf{v}^{A0}$ $\mathbf{v}^{B0}$. At some instant of time, the two particles collide. At the point of collision, the centers of the spheres have position vectors $\mathbf{y}^{A}$ $\mathbf{y}^{B}$. The restitution coefficient for the contact is denoted by *e*. Find a formula for the velocities of the spheres after impact. Hence, deduce an expression for the change in kinetic energy during the impact.

This is a straightforward vector algebra exercise. We have two unknown velocity vectors: $\mathbf{v}^{A1}$ $\mathbf{v}^{B1}$, and two vector equations – momentum conservation, and the restitution coefficient formula.

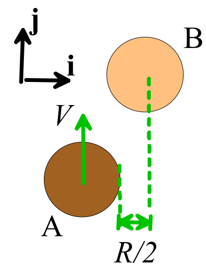


Calculation

1. Note that a unit vector normal to the tangent plane can be calculated from the position vectors of the centers at the impact as $\mathbf{n} = \left(\mathbf{y}^{A} - \mathbf{y}^{B}\right)/\left|\mathbf{y}^{A} - \mathbf{y}^{B}\right|$. It doesn't matter whether you choose to take $\mathbf{n}$ to point from A to B or the other way around – the formula will work either way.
2. Momentum conservation requires that
$$m_B\ \mathbf{v}^{B1} + m_A\mathbf{v}^{A1} = m_B\ \mathbf{v}^{B0} + m_A\mathbf{v}^{A0}$$
3. The restitution coefficient formula gives
$$\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right) = \left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right) - (1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$
4. We can solve (2) and (3) for $\mathbf{v}^{B1}$ by multiplying (3) by m_A and adding the equations, which gives
$$\left(m_B + m_A\right)\mathbf{v}^{B1} = m_B\ \mathbf{v}^{B0} + m_A\mathbf{v}^{A0} + m_A\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right) - (1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}\right]$$
$$\Rightarrow \mathbf{v}^{B1} = \mathbf{v}^{B0} - \frac{m_A}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$
5. Similarly, we can solve for $\mathbf{v}^{A1}$ by multiplying (3) by m_B and subtracting (3) from (2), with the result
$$\mathbf{v}^{A1} = \mathbf{v}^{A0} + \frac{m_B}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$
6. The change in kinetic energy during the collision can be calculated as
$$\Delta T = \frac{m_A}{2}\mathbf{v}^{A1}\cdot\mathbf{v}^{A1} + \frac{m_B}{2}\mathbf{v}^{B1}\cdot\mathbf{v}^{B1} - \left(\frac{m_A}{2}\mathbf{v}^{A0}\cdot\mathbf{v}^{A0} + \frac{m_B}{2}\mathbf{v}^{B0}\cdot\mathbf{v}^{B0}\right)$$
7. Substituting the results of (4) and (5) for $\mathbf{v}^{B1}$ and $\mathbf{v}^{A1}$ and simplifying the result gives
$$\Delta T = \frac{m_A m_B\left(e^2 - 1\right)}{2\left(m_A + m_B\right)}\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]^2$$

Note that the energy change is zero if *e=1* (perfectly elastic collisions) and is always negative for *e<1* (i.e. the kinetic energy after collision is less than that before collision).

Example 3: This is just a boring example to help illustrate the practical application of the vector formulas in the preceding example. In the figure shown, disk A has a vertical velocity V at time $t=0$, while disk B is stationary. The two disks both have radius R, have the same mass, and the restitution coefficient between them is e. Gravity can be neglected. Calculate the velocity vector of each disk after collision.

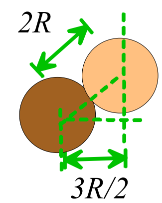


Calculation

1. Before impact, the velocity vectors are $\mathbf{v}^{A0} = V\mathbf{j} \quad \mathbf{v}^{B0} = \mathbf{0}$
2. A unit vector parallel to the line joining the two centers is $\mathbf{n} = (3\mathbf{i} + \sqrt{7}\mathbf{j})/4$ (to see this, apply Pythagoras theorem to the triangle shown in the figure).
3. The velocities after impact are

$$\mathbf{v}^{B1} = \mathbf{v}^{B0} - \frac{m_A}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

$$\mathbf{v}^{A1} = \mathbf{v}^{A0} + \frac{m_B}{m_B + m_A}(1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

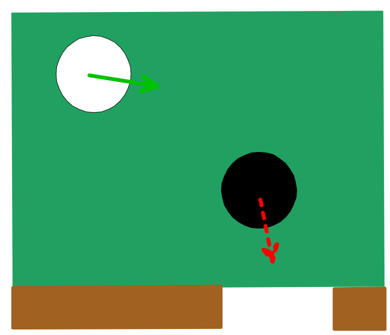


Substituting the vectors gives

$$\mathbf{v}^{B1} = -\frac{1}{2}(1+e)\left[\left(-V\mathbf{j}\right)\cdot(3\mathbf{i} + \sqrt{7}\mathbf{j})/4\right](3\mathbf{i} + \sqrt{7}\mathbf{j})/4 = \frac{1+e}{32}V(3\sqrt{7}\mathbf{i} + 7\mathbf{j})$$

$$\mathbf{v}^{A1} = V\mathbf{j} + \frac{1}{2}(1+e)\left[\left(-V\mathbf{j}\right)\cdot(3\mathbf{i} + \sqrt{7}\mathbf{j})/4\right](3\mathbf{i} + \sqrt{7}\mathbf{j})/4 = -\frac{3\sqrt{7}(1+e)}{32}V\mathbf{i} + \frac{25-7e}{32}V\mathbf{j}$$

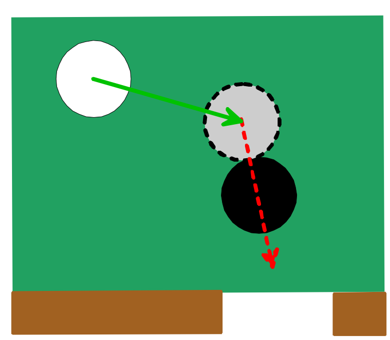
Example 4: How to play pool (or snooker, billiards, or your own favorite bar game involving balls, a stick, and a table…). The figure shows a typical problem faced by a pool player – where should the queue ball hit the eight ball to send it into the pocket?



This is easily solved – the eight ball is stationary before impact, and after impact has a velocity

$$\mathbf{v}^{B1} = \frac{1}{2}(1+e)\left[\mathbf{v}^{A0}\cdot\mathbf{n}\right]\mathbf{n}$$

Notice that the velocity is parallel to the unit vector **n.** This vector is parallel to a line connecting the centers of the two balls at the instant of impact. So the correct thing to do is to visualize an imaginary ball just touching the eight ball, in line with the pocket, and aim the queue ball at the imaginary ball. Easy!

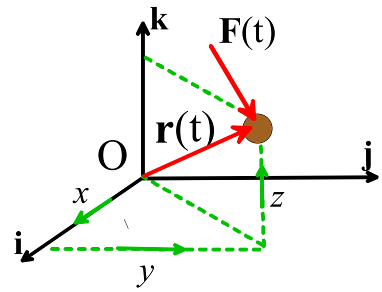


The real secret to being a successful pool player is not potting the balls – that part is easy. It is controlling where the queue ball goes after impact. You may have seen experts make a queue ball reverse its direction after an impact (appearing to bounce off the stationary ball); or make the queue ball follow the struck ball after the impact. According to the simple equations developed here, this is impossible – but a pool table is more complicated, because the balls rotate, and are in contact with a table. By giving the queue ball *spin*, an expert player can move the queue ball around at will. To make the ball rebound, it must be struck low down (below the 'center of percussion') to give it a reverse spin; to make it follow the struck ball, it should be struck high up, to make it roll towards the ball to be potted. Giving the ball a sideways spin can make it rebound in a controllable direction laterally as well. And it is even possible to make a queue ball travel in a *curved* path with the right spin.

Never let it be said that you don't learn useful skills in engineering classes!

4.3 Angular impulse-momentum relations

4.3.1 Definition of the angular impulse of a force



The angular impulse of a force is the time integral of the *moment* exerted by the force.

To make the concept precise, consider a particle that is subjected to a time varying force $\mathbf{F}(t)$, with components in a fixed basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$, then

$$\mathbf{F}(t) = F_x(t)\mathbf{i} + F_y(t)\mathbf{j} + F_z(t)\mathbf{k}$$

Let

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

denote the position vector of the particle, and

$$\mathbf{M}(t) = \mathbf{r}(t)\times\mathbf{F}(t) = \left(y(t)F_z(t) - z(t)F_y(t)\right)\mathbf{i} + \left(z(t)F_x(t) - x(t)F_z(t)\right)\mathbf{j} + \left(x(t)F_y(t) - y(t)F_x(t)\right)\mathbf{k}$$

the moment of the force about the origin.

The ***Angular Impulse*** exerted by the force about O during a time interval $t_0 \le t \le t_1$ is defined as

$$\mathbf{A} = \int_{t_0}^{t_1} \mathbf{M}(t)dt = \int_{t_0}^{t_1} \mathbf{r}(t)\times\mathbf{F}(t)dt$$

The angular impulse is a vector, and can be expressed as components in a basis

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

$$A_x = \int_{t_0}^{t_1} \left(y(t)F_z(t) - z(t)F_y(t)\right)dt$$

$$A_y = \int_{t_0}^{t_1} \left(z(t)F_x(t) - x(t)F_z(t)\right)dt$$

$$A_z = \int_{t_0}^{t_1} \left(x(t)F_y(t) - y(t)F_x(t)\right)dt$$

If you know the moment as a function of time, you can calculate its angular impulse using simple calculus. For example for a *constant* moment, with vector value $\mathbf{M}_0$, the impulse is $\mathbf{A} = \mathbf{M}_0(t_1 - t_0)$

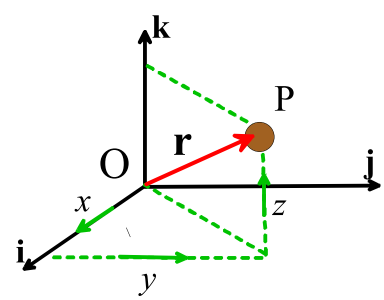
4.3.2 Definition of the angular momentum of a particle

The angular momentum of a particle is simply the cross product of the particle's position vector with its linear momentum

$$\mathbf{h} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

The angular momentum is a vector – the direction of the vector is perpendicular to its velocity and its position vectors.

4.3.3 Angular impulse – Angular Momentum relation for a single particle



- Consider a particle that is subjected to a force $\mathbf{F}(t)$ for a time interval $t_0 \le t \le t_1$.
- Let $\mathbf{r}(t)$ denote the position vector of the particle
- Let $\mathbf{M}(t) = \mathbf{r}(t) \times \mathbf{F}(t)$ denote the moment of **F** about the origin
- Let **A** denote the angular impulse exerted on the particle
- Let $\mathbf{h}_0$ denote the angular momentum at time t_0
- Let $\mathbf{h}_1$ denote the angular momentum at time t_1

The momentum conservation equation can be expressed either in differential or integral form.

1. In differential form

$$\mathbf{M} = \frac{d\mathbf{h}}{dt}$$

2. In integral form

$$\mathbf{A} = \mathbf{h}_1 - \mathbf{h}_0$$

Proof: Although it's not obvious, these are just another way of writing Newton's laws of motion. To show this, we'll derive the differential form. Start with Newton's law

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt}$$

Take the cross product of both sides with **r**

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\frac{d\mathbf{v}}{dt}$$

Note that $\mathbf{v} \times m\mathbf{v} = \mathbf{0}$ since the cross product of two parallel vectors is zero. We can add this to the right hand side, which shows that

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times m\frac{d\mathbf{v}}{dt} + \mathbf{v} \times m\mathbf{v} = \mathbf{r} \times m\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{r}}{dt} \times m\mathbf{v} = \frac{d}{dt}\left(\mathbf{r} \times m\mathbf{v}\right) = \frac{d\mathbf{h}}{dt}$$

This yields the required relation.

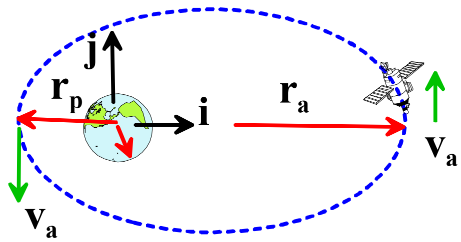
Angular momentum conservation: For the special case where the force is *parallel* to **r**, the moment of the force acting on the particle is zero ($\mathbf{r} \times \mathbf{F} = \mathbf{0}$), and angular momentum is constant

$$\frac{d}{dt}\left(\mathbf{r}\times m\mathbf{v}\right)=\mathbf{0}$$

4.3.4 Examples using Angular Impulse – Angular Momentum relations for a single particle

The angular impulse-angular momentum equations are particularly helpful when you need to solve problems where particles are subjected to a single force, which acts through a fixed point. They can also be used to analyze rotational motion of a massless frame containing one or more particles.

Example 1: Orbital motion. A satellite is launched into a geostationary transfer orbit by the ARIANE V launch facility. At its perigee (the point where the satellite is closest to the earth) the satellite has speed 10.2km/sec and altitude 250km. At apogee (the point where the satellite is furthest from the earth) the satellite has altitude 35950 km. Calculate the speed of the satellite at apogee.



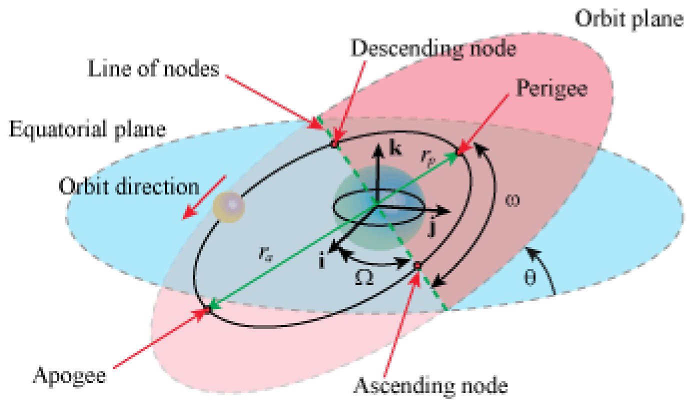
Assumptions:

1. We assume that the only force acting on the satellite is the gravitational attraction of the earth
2. The earth's radius is 6378.145km

Calculation:

1. Since the gravitational force on the satellite always acts towards the center of the earth, angular momentum about the earth's center is conserved.
2. At both perigee and apogee, the velocity vector of the satellite must be perpendicular to its position vector. To see this, note that at the point where the satellite is closest and furthest from the earth, the distance to the earth is a max or min, and so the derivative of the distance of the satellite from the earth must vanish, i.e.
$$\frac{d}{dt}\left(|\mathbf{r}|\right)=\frac{d}{dt}\left(\sqrt{\mathbf{r}\cdot\mathbf{r}}\right)=\mathbf{0} \qquad \Rightarrow \frac{1}{2}\frac{1}{\sqrt{\mathbf{r}\cdot\mathbf{r}}}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}\right)=0\Rightarrow \mathbf{r}\cdot\mathbf{v}=0$$
where we have used the chain rule to evaluate the time derivative of $\sqrt{\mathbf{r}\cdot\mathbf{r}}$. Recall that if the dot product of two vectors vanishes, they are mutually perpendicular. We take a coordinate system with **i** and **j** in the plane of the orbit, and **k** perpendicular to the orbit.
3. We take the satellite orbit to lie in the **i** , **j** plane with **k** perpendicular to the orbit. The angular momentum at apogee and perigee is then
$$\mathbf{h}_p=\mathbf{r}_p\times m\mathbf{v}_p=r_p m v_p\mathbf{k}$$
$$\mathbf{h}_a=\mathbf{r}_a\times m\mathbf{v}_a=r_a m v_a\mathbf{k}$$
where r_a, r_p are the distance of the satellite from the center of the earth at apogee and perigee, and v_a, v_p are the corresponding speeds.
4. Since angular momentum is conserved it follows that
$$\mathbf{h}_p=\mathbf{h}_a\Rightarrow r_p m v_p=r_a m v_a \;\Rightarrow v_a=r_p v_p/r_a$$
5. Substituting numbers yields 1.6 km/s

Example 2: More orbital motion. The orbit for a satellite is normally specified by a set of angles specifying the inclination of the orbit, and by quoting the distance of the satellite from the earths center at apogee and perigee r_a, r_p. It is possible to calculate the speed of the satellite at perigee and apogee from this information.



Calculation

1. From the previous example, we know that the distances and velocities are related by

$$r_p m v_p = r_a m v_a$$

2. The system is conservative, so the total energy of the system is conserved.
3. The potential energies when the satellite is at perigee and apogee are

$$V_p = -\frac{GMm}{r_p} \qquad V_a = -\frac{GMm}{r_a}$$

4. The kinetic energies of the satellite at perigee and apogee are

$$T_p = \frac{1}{2} m v_p^2 \qquad T_a = \frac{1}{2} m v_a^2$$

5. The kinetic energy of the earth can be assumed to be constant. Energy conservation therefore shows that

$$\frac{1}{2} m v_p^2 - \frac{GMm}{r_p} = \frac{1}{2} m v_a^2 - \frac{GMm}{r_a}$$

6. The results of (1) and (5) give two equations that can be solved for v_a, v_p in terms of known parameters. For example, (1) shows that $v_a = v_p r_p / r_a$ - this can be substituted into (5) to see that

$$\frac{1}{2} m v_p^2 - \frac{GMm}{r_p} = \frac{1}{2} m v_p^2 \frac{r_p^2}{r_a^2} - \frac{GMm}{r_a}$$

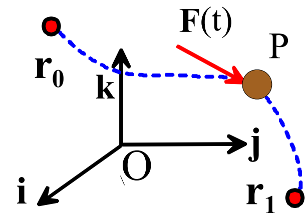
$$\Rightarrow v_p^2 \frac{r_a^2 - r_p^2}{r_a^2} = 2GM \left(\frac{1}{r_p} - \frac{1}{r_a} \right) \qquad \Rightarrow v_p = \sqrt{\frac{2GMr_a}{r_p (r_a + r_p)}}$$

Similarly, at apogee

$$v_a = \sqrt{\frac{2GMr_p}{r_a (r_a + r_p)}}$$

4.4 Summary of definitions and equations

Work, Power, Kinetic Energy



- The *Power* developed by a force, (or the *rate of work done* by the force) is

$$P = \mathbf{F} \cdot \mathbf{v} .$$

- The *work done* by the force during a time interval $t_0 \le t \le t_1$ is

$$W = \int_{t_0}^{t_1} P dt = \int_{t_0}^{t_1} \mathbf{F} \cdot \mathbf{v} dt$$

The work done by the force can also be calculated by integrating the force vector along the path traveled by the force, as

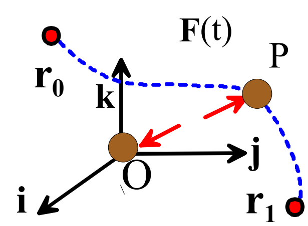
$$W = \int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{r}_0, \mathbf{r}_1$ are the initial and final positions of the force.

- The *Kinetic Energy* of a particle is

$$T = \frac{1}{2} m |\mathbf{v}|^2 = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)$$

Conservative forces and potential energy



A force (or pair of forces) is *conservative* if the work done by the force when it moves between any two points is the same for all paths joining the two points

The ***potential energy of a conservative force*** is defined as the negative of the work done by the force in moving from some arbitrary initial position $\mathbf{r}_0$ to a new position $\mathbf{r}$, i.e.

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} + \text{constant}$$

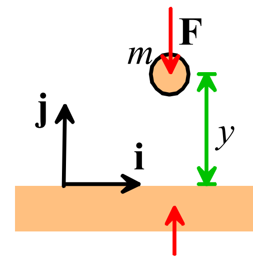
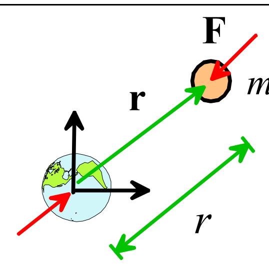
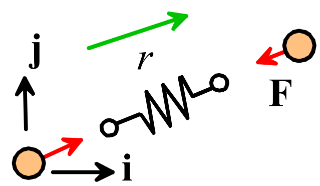
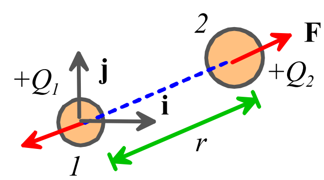
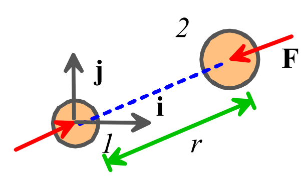
Alternatively

$$\mathbf{F} = -\text{grad}(V)$$

$$F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k} = -\left(\frac{\partial V}{\partial x} \mathbf{i} + \frac{\partial V}{\partial y} \mathbf{j} + \frac{\partial V}{\partial z} \mathbf{k} \right)$$

Table of potential energy relations

In practice, however, we rarely need to do the integrals to calculate the potential energy of a force, because there are very few different kinds of force. For most engineering calculations the potential energy formulas listed in the table below are sufficient.

Type of force	Force vector	Potential energy	
Gravity acting on a particle near earths surface	$\mathbf{F} = -mg\mathbf{j}$	$V = mgy$	
Gravitational force exerted on mass m by mass M at the origin	$\mathbf{F} = -\dfrac{GMm}{r^3}\mathbf{r}$	$V = -\dfrac{GMm}{r}$	
Force exerted by a spring with stiffness k and unstretched length L_0	$\mathbf{F} = -k(r - L_0)\dfrac{\mathbf{r}}{r}$	$V = \dfrac{1}{2}k\left(r - L_0\right)^2$	
Force acting between two charged particles	$\mathbf{F} = \dfrac{Q_1 Q_2}{4\pi\varepsilon r^3}\mathbf{r}$	$V = \dfrac{Q_1 Q_2}{4\pi\varepsilon r}$	
Force exerted by one molecule of a noble gas (e.g. He, Ar, etc) on another (Lennard Jones potential). a is the equilibrium spacing between molecules, and E is the energy of the bond.	$\mathbf{F} = -12\dfrac{E}{a}\left[2\left(\dfrac{a}{r}\right)^{13} - \left(\dfrac{a}{r}\right)^{7}\right]\dfrac{\mathbf{r}}{r}$	$E\left[\left(\dfrac{a}{r}\right)^{12} - 2\left(\dfrac{a}{r}\right)^{6}\right]$	

Power-Work-kinetic energy relations for a single particle

The *Power-kinetic energy* relation for the particle states that the rate of work done by **F** is equal to the rate of change of kinetic energy of the particle, i.e.

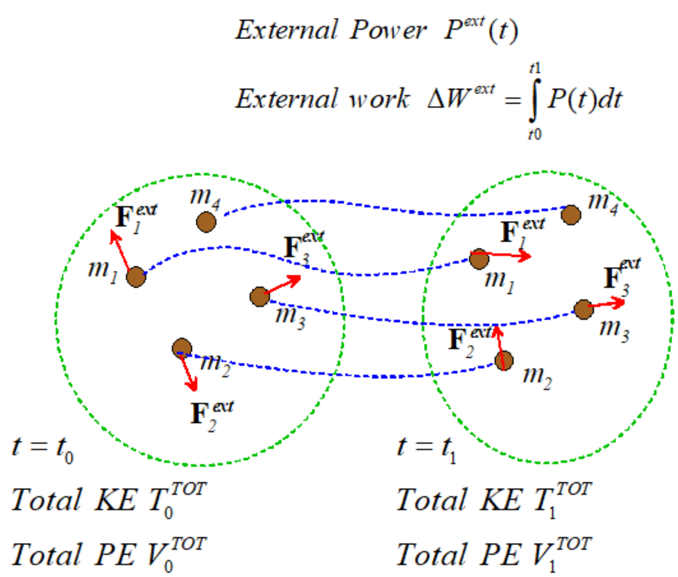
$$P = \frac{dT}{dt}$$

The *Work-kinetic energy* relation for a particle says that the total work done by the force **F** on the particle is equal to the change in the kinetic energy of the particle.

$$W = T - T_0$$

Power-Work-kinetic energy relations for a conservative system of particles

A system is conservative if a potential energy can be defined for all internal forces in the system



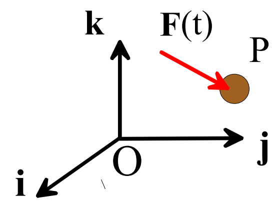
1. At some time t_0 the system has and kinetic energy T_0^{TOT}
2. At some later time t_1 the system has kinetic energy T_1^{TOT} .
3. Let V_0^{TOT} denote the potential energy of the internal forces at time t_0
4. Let V_1^{TOT} denote the potential energy of the internal forces at time t
5. Let ΔW^{ext} denote the total work done on the system between $t_0 \le t \le t_1$

Power Energy Relation: $P^{ext} = \frac{d}{dt}\left(T_1^{TOT} + V_1^{TOT}\right)$

Work Energy Relation: $\Delta W^{ext} = T_1^{TOT} + V_1^{TOT} - \left(T_0^{TOT} + V_0^{TOT}\right)$

Energy conservation law $\Delta W^{ext} = 0 \quad \Rightarrow T_1^{TOT} + V_1^{TOT} = \left(T_0^{TOT} + V_0^{TOT}\right)$

Linear impulse of a force



$$\Im = \int_{t_0}^{t_1} \mathbf{F}(t)dt$$

Linear momentum of a particle

The linear momentum of a particle is simply the product of its mass and velocity

$$\mathbf{p} = m\mathbf{v}$$

The linear momentum is a vector – its direction is parallel to the velocity of the particle.

Impulse-momentum relations for a single particle

- Consider a particle that is subjected to a force $\mathbf{F}(t)$ for a time interval $t_0 \le t \le t_1$.
- Let $\Im$ denote the impulse exerted by **F** on the particle
- Let $\mathbf{p}_0$ denote the linear momentum of the particle at time t_0 .
- Let $\mathbf{p}_1$ denote the linear momentum of the particle at time t_1 .

In differential form

$$\mathbf{F}(t) = \frac{d\mathbf{p}}{dt}$$

In integral form

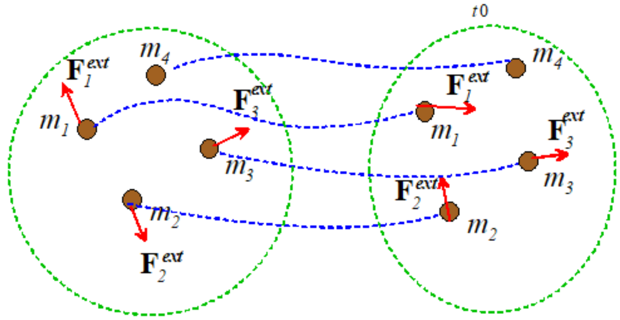
$$\Im = \mathbf{p}_1 - \mathbf{p}_0$$

This is the integral of Newton’s law of motion with respect to time.

Impulse-momentum relations for a system of particles

$$Total\ \ External\ \ Force\ \ \mathbf{F}^{TOT}(t)$$

$$Total\ External\ Impulse\ \ \Im^{TOT} = \int_{t0}^{t1} \mathbf{F}^{TOT}(t)dt$$



$t = t_0$ $\qquad$ $t = t_1$

$Total\ \ momentum\ \ \mathbf{p}_0^{TOT}$ $\qquad$ $Total\ \ momentum\ \ \mathbf{p}_1^{TOT}$

In differential form $\sum_{particles} \mathbf{F}_i^{ext}(t) = \frac{d\mathbf{p}^{TOT}}{dt}$

In integral form $\Im^{TOT} = \mathbf{p}_1^{TOT} - \mathbf{p}_0^{TOT}$

If *no* external forces act on a system of particles, *their total linear momentum is conserved,* i.e.

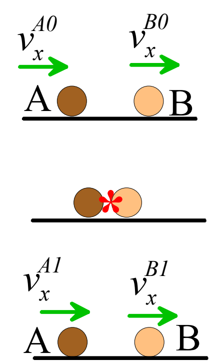
$$\mathbf{p}_1^{TOT} = \mathbf{p}_0^{TOT}$$

Collisions

If no external impulse acts on two particles as they collide, their total momentum is conserved.

To calculate velocities after impact we define the *restitution coefficient*

Restitution coefficient for straight line motion



$\mathbf{v}^{A0} = v_x^{A0}\mathbf{i}$, $\mathbf{v}^{B0} = v_x^{B0}\mathbf{i}$ denote the velocities of A and B just before the collision

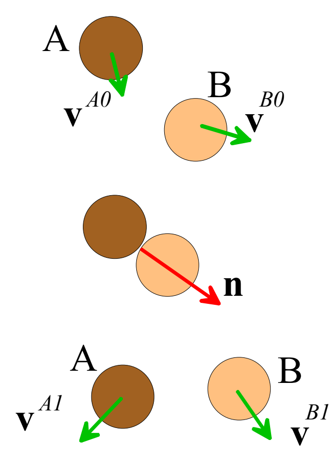
$\mathbf{v}^{A1} = v_x^{A1}\mathbf{i}$, $\mathbf{v}^{B1} = v_x^{B1}\mathbf{i}$ denote the velocities of A and B just after the collision.

The velocities before and after impact are related by

$$v^{B1} - v^{A1} = -e\left(v^{B0} - v^{A0}\right)$$

where e is the restitution coefficient.

Restitution coefficient for 3D frictionless collisions



Define

$\mathbf{v}^{A0}$ $\mathbf{v}^{B0}$ to denote velocities of the particles before collision

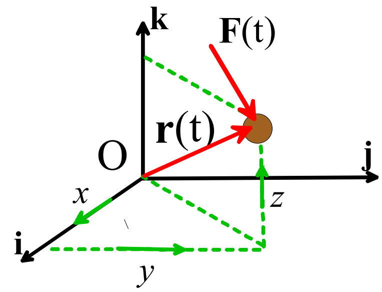
$\mathbf{v}^{A1}$ $\mathbf{v}^{B1}$ to denote velocities of the particles after collision

In addition, we let $\mathbf{n}$ be a unit vector normal to the common tangent plane at the point of contact (if the two colliding particles are spheres or disks the vector is parallel to the line joining their centers).

The velocities before and after collision are related by

$$\left(\mathbf{v}^{B1} - \mathbf{v}^{A1}\right) = \left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right) - (1+e)\left[\left(\mathbf{v}^{B0} - \mathbf{v}^{A0}\right)\cdot\mathbf{n}\right]\mathbf{n}$$

Angular impulse



The ***Angular Impulse*** exerted by the force about O during a time interval $t_0 \le t \le t_1$ is defined as

$$\mathbf{A} = \int_{t_0}^{t_1} \mathbf{M}(t) dt = \int_{t_0}^{t_1} \mathbf{r}(t) \times \mathbf{F}(t) dt$$

Angular momentum of a particle

$$\mathbf{h} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v}$$

Angular impulse – Angular Momentum relation for a single particle

- Consider a particle that is subjected to a force $\mathbf{F}(t)$ for a time interval $t_0 \le t \le t_1$.
- Let $\mathbf{r}(t)$ denote the position vector of the particle
- Let $\mathbf{M}(t) = \mathbf{r}(t) \times \mathbf{F}(t)$ denote the moment of **F** about the origin
- Let **A** denote the angular impulse exerted on the particle
- Let $\mathbf{h}_0$ denote the angular momentum at time t_0
- Let $\mathbf{h}_1$ denote the angular momentum at time t_1

In differential form $\mathbf{M} = \dfrac{d\mathbf{h}}{dt}$

In integral form $\mathbf{A} = \mathbf{h}_1 - \mathbf{h}_0$