Chapter 6
Rigid Body Dynamics

6.1 Introduction

In practice, it is often not possible to idealize a system as a particle. In this section, we construct a more sophisticated description of the world, in which objects *rotate*, in addition to translating. This general branch of physics is called 'Rigid Body Dynamics.'

Rigid body dynamics has many applications. In vehicle dynamics, we are often more worried about controlling the orientation of our vehicle than its path – an aircraft must keep its shiny side up, and we don't want a spacecraft tumbling uncontrollably. Rigid body mechanics is used extensively to design power generation and transmission systems, from jet engines, to the internal combustion engine, to gearboxes. A typical problem is to convert rotational motion to linear motion, and vice-versa. Rigid body motion is also of great interest to people who design prosthetic devices, implants, or coach athletes: here, the goal is to understand human motion, to protect athletes from injury or improve their performance, or to design devices that replicate the complicated motion of a human joint correctly. For example, Professor Crisco's orthopaedics lab at Brown studies human motion and the forces they generate at human joints, to help understand how injuries occur and how they can be prevented.

The motion of a rigid body is often very counter-intuitive. That's why there are so many toys that exploit the properties of rigid bodies: the motion of a spinning top; a boomerang; the 'rattleback' and a Frisbee can all be explained using the equations derived in this section.

Here is a quick outline of how we analyze motion of rigid bodies.

1. A rigid body is idealized as an infinite number of small particles, connected by two-force members.
2. We already know the equations of motion for a system of particles (Section 4 of the notes):

   The force-momentum equation $\sum_i \mathbf{F}_i^{ext} = \frac{d\mathbf{p}}{dt} = \frac{d}{dt}\sum_{i=1}^{N} m_i \mathbf{v}_i$

   The moment – angular momentum equation $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = \frac{d\mathbf{h}}{dt} = \frac{d}{dt}\sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i$

   The work-kinetic energy equation $\sum_i \mathbf{F}_i^{ext} \cdot \mathbf{v}_i = \frac{dT}{dt} = \frac{d}{dt}\sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i$

3. These equations tell us how a rigid body moves. But to use them, we would need to keep track track of an infinite number of particles! To simplify the problem, we set up some mathematical methods that allow us to express the position and velocity of every point in a rigid body in terms of the position $\mathbf{r}_G$, velocity $\mathbf{v}_G$ and acceleration $\mathbf{a}_G$ of its center of mass, and its rotation tensor $\mathbf{R}$(quantifying its orientation) and its angular velocity $\boldsymbol{\omega}$, and angular acceleration $\boldsymbol{\alpha}$. This allows us to write the linear momentum, angular momentum, and kinetic energy of a rigid body in the form

$$\mathbf{p} = M\mathbf{v}_G \qquad \mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega} \qquad T = \frac{1}{2}M\mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_G\boldsymbol{\omega}$$

   where M is the total mass of the body and $\mathbf{I}_G$ is its mass moment of inertia.

4. We can then derive the rigid body equations of motion:

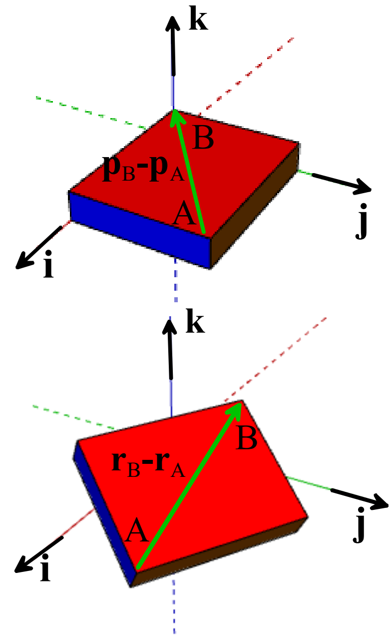
$$\sum_i \mathbf{F}_i^{ext} = M\mathbf{a}_G \qquad \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = M\mathbf{r}_G \times \mathbf{a}_G + \mathbf{I}_G\boldsymbol{\alpha} + \boldsymbol{\omega} \times \left[\mathbf{I}_G\boldsymbol{\omega}\right]$$

6.2 Describing Motion of a Rigid Body

We describe motion of a particle using its position, velocity and acceleration. We can describe the *position* of a rigid body in the same way - we could specify the position, velocity and acceleration of any convenient point in the body (we usually use the center of mass). But we also need a way to describe the *orientation* of a rigid body, and its rotational motion.

In this section, we define the various mathematical quantities that we use to describe rotation, angular velocity, and angular acceleration.

6.2.1 Describing rotations: The Rotation Tensor (or matrix)



Rotations are quantified by a mathematical object called a **rotation tensor**. It is defined as follows:

1. Choose some convenient initial orientation of the rigid body (eg for the rectangular prism in the figure, we chose to make the faces perpendicular to the $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ directions.
2. When the body is rotated, every line in the body (eg the sides) moves to a new orientation, without changing its length. We can describe this orientation change as a *mapping*. Let A and B be two arbitrary points in the body. Let $\mathbf{p}_A,\mathbf{p}_B$ be the initial positions of these points, and let $\mathbf{r}_A,\mathbf{r}_B$ be their final positions. We introduce the 'rotation tensor[1]' **R** which has the property that
$$\mathbf{r}_B-\mathbf{r}_A=\mathbf{R}(\mathbf{p}_B-\mathbf{p}_A)$$

When we solve problems, we always express vectors as components in some basis. When we do this, **R** becomes a matrix. For example, if

$$\mathbf{p}_B-\mathbf{p}_A=x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k} \qquad \mathbf{r}_B-\mathbf{r}_A=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

we would write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{yz} & R_{zy} & R_{zz} \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

Here, $R_{11},R_{12},...$ are a set of nine numbers (or sometimes formulas). Following the usual rules of matrix-vector multiplication, this is just a short-hand notation for

$$x=R_{xx}x_0+R_{xy}y_0+R_{xz}z_0$$
$$y=R_{yx}x_0+R_{yy}y_0+R_{yz}z_0$$
$$z=R_{zx}x_0+R_{zy}y_0+R_{zz}z_0$$

The subscripts on **R** are meant to you help remember what each element in the matrix does – for example, R_{xx} maps the x_0 onto *x*, R_{xy} maps the y_0 onto *x*, and so on.

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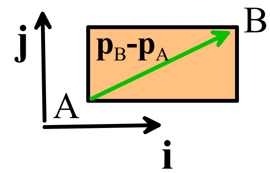
[1] By definition, a 'second order tensor' maps a vector onto another vector. In actual calculations **R** is always just a matrix, but 'tensor' sounds better.

So when we solve a problem, how do we go about finding R? Let me count the ways:

Rotations in two dimensions:

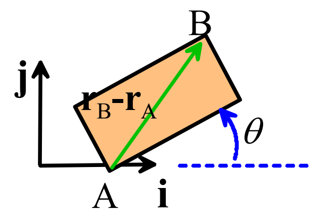


Life is simple in 2D. In this case our rigid body must lie in the **i,j** plane, so we can only rotate it about an axis parallel to the **k** direction. A counter-clockwise rotation through an angle θ about the **k** axis is produced by[2]

$$\mathbf{R}=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$$

For example, a vector $L\mathbf{i}$ that start parallel to the **i** axis is mapped to

$$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}L \\ 0\end{bmatrix}=\begin{bmatrix}L\cos\theta \\ L\sin\theta\end{bmatrix}=L\cos\theta\mathbf{i}+L\sin\theta\mathbf{j}$$



Rotation about a known axis

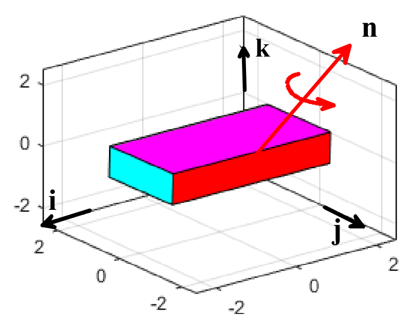
3D is a bit more difficult. Any rotation can always be expressed as a rotation through some angle θ about some axis parallel to a unit vector **n** **(we always use the right hand screw convention**). In some problems you can see what **n** and θ are: then you can write down a unit vector parallel to **n**

$$\mathbf{n}=n_x\mathbf{i}+n_y\mathbf{j}+n_z\mathbf{k}$$

and then use the 'Rodriguez Formula'

$$\mathbf{R}=\begin{bmatrix}\cos\theta+(1-\cos\theta)n_x^2 & (1-\cos\theta)n_xn_y-\sin\theta n_z & (1-\cos\theta)n_xn_z+\sin\theta n_y \\ (1-\cos\theta)n_xn_y+\sin\theta n_z & \cos\theta+(1-\cos\theta)n_y^2 & (1-\cos\theta)n_yn_z-\sin\theta n_x \\ (1-\cos\theta)n_xn_z-\sin\theta n_y & (1-\cos\theta)n_yn_z+\sin\theta n_x & \cos\theta+(1-\cos\theta)n_z^2\end{bmatrix}$$

(This formula is impossible to remember – that's what Google is for).



If you are given a rotation matrix **R**, and need to find **n** and θ , you can use the formulas:

$$1+2\cos\theta=R_{xx}+R_{yy}+R_{zz}$$

$$\mathbf{n}=\frac{1}{2\sin\theta}\left[\left(R_{zy}-R_{yz}\right)\mathbf{i}+\left(R_{xz}-R_{zx}\right)\mathbf{j}+\left(R_{yx}-R_{xy}\right)\mathbf{k}\right]$$

The second formula blows up if $\sin(\theta)=0$. If θ is zero or 2π you can simply set $\mathbf{R}=\mathbf{1}$ (the identity), and **n** can be anything you like. For $\theta=\pi$ you can use

$$\mathbf{n}=\sqrt{\frac{R_{xx}-\cos\theta}{1-\cos\theta}}\mathbf{i}\pm\sqrt{\frac{R_{yy}-\cos\theta}{1-\cos\theta}}\mathbf{j}\pm\sqrt{\frac{R_{zz}-\cos\theta}{1-\cos\theta}}\mathbf{k}$$

The signs of the square roots have to be chosen so that $n_xn_y=R_{xy}/2 \quad n_xn_z=R_{xz}/2 \quad n_yn_z=R_{yz}/2$

[2] (Tip: it's easy to remember this but it's hard to remember where to put the negative sign. You can always figure this out by noting that a 90 degree counter-clockwise rotation maps a vector parallel to the **i** direction onto a vector parallel to the **j** direction.)

In robotics, game engines, and vehicle dynamics the axis-angle representation of a rotation is often stored as a *quaternion*. We won't use that here, but mention it in passing in case you come across it in practice. A quaternion is four numbers $[q_0, q_x, q_y, q_x]$ that are related to $\mathbf{n}$ and θ through the formulas:

$$q_0 = \cos(\theta/2)$$
$$q_x = n_x \sin(\theta/2) \qquad q_y = n_y \sin(\theta/2) \qquad q_z = n_z \sin\left(\theta/2\right)$$

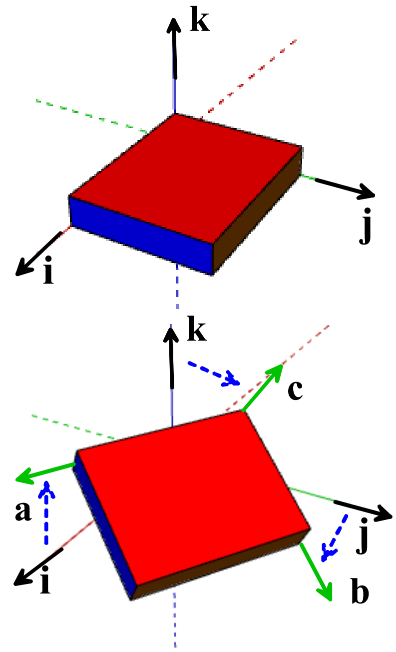
Mapping the coordinate axes

In some problems we might know what happens to vectors that are parallel to the {**i,j,k**} directions in the initial rigid body (eg we might know what happens to the sides of our rectangular prism). For example, we might know that $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ map to **(unit)** vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. In that case we can write down each of $\mathbf{a},\mathbf{b},\mathbf{c}$ as components in $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \qquad \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k} \qquad \mathbf{c} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$$

and use the formula

$$\mathbf{R} = \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$



A sequence of rotations

Suppose we rotate an object twice (perhaps about two different axes). How do we describe the result of two rotations? That's not hard. Suppose we do the first rotation with one mapping

$$\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}^{(1)}(\mathbf{p}_B - \mathbf{p}_A)$$

Now we rotate our body again – this maps $\mathbf{r}_B - \mathbf{r}_A$ onto some new vector $\mathbf{u}_B - \mathbf{u}_A$:

$$(\mathbf{u}_B - \mathbf{u}_A) = \mathbf{R}^{(2)}(\mathbf{r}_B - \mathbf{r}_A)$$

We can therefore write

$$(\mathbf{u}_B - \mathbf{u}_A) = \mathbf{R}^{(2)}\mathbf{R}^{(1)}(\mathbf{p}_B - \mathbf{p}_A)$$

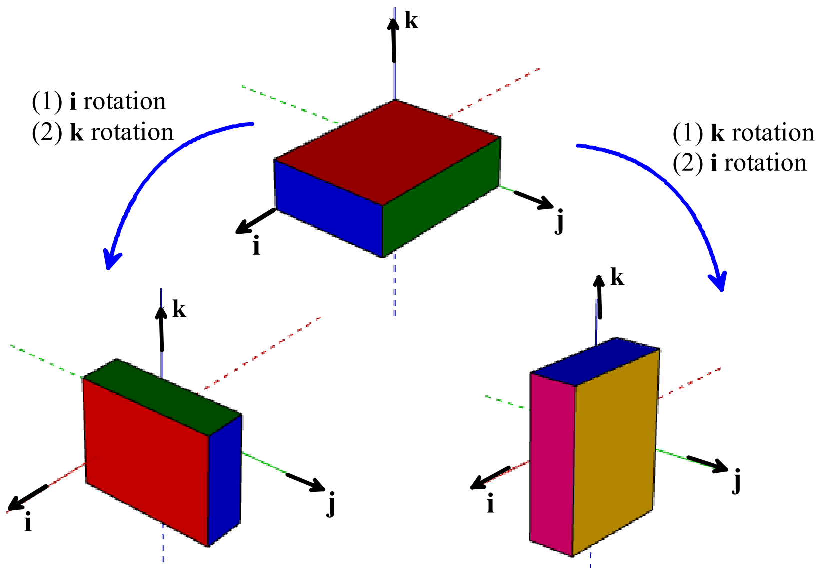
We see that **Sequential rotations are matrix products**

$$\mathbf{R} = \mathbf{R}^{(2)}\mathbf{R}^{(1)}$$

Health warning: Matrix products (and hence sequences of rotations) do not commute

$$\mathbf{R}^{(1)}\mathbf{R}^{(2)} \neq \mathbf{R}^{(2)}\mathbf{R}^{(1)}$$

For example, the figure below shows the change in orientation caused by (a) a 90 degree positive rotation about **i** followed by a 90 degree positive rotation about **k** (the figure on the left); and (b) a 90 degree positive rotation about **k** followed by a 90 degree positive rotation about **i** (the figure on the right).



Orthogonality of R

The rotation tensor (matrix) has a very important property:

If you multiply **R** *by its transpose, the result is always the identity matrix.*

Another way to say this is that

The transpose of **R** *is equal to its inverse*

Let's try this with the 2D rotation matrix

$$\mathbf{R}\mathbf{R}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{R}^T\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A matrix or tensor with this property is said to be *orthogonal*.

Why is this? It turns out that *a length-preserving mapping must be an orthogonal tensor*. To see this, let's calculate the length of the rotated vector $\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)$. We need to remember two vector/matrix operations:

1. We can calculate the length of a vector by dotting it with itself and taking the square root
2. For a vector **u** and a matrix **R,** we know (or can show!) that $(\mathbf{R}\mathbf{u})\cdot(\mathbf{R}\mathbf{u}) = \mathbf{u}\cdot\left(\mathbf{R}^T\mathbf{R}\mathbf{u}\right)$

This means

$$\sqrt{(\mathbf{r}_B - \mathbf{r}_A)\cdot(\mathbf{r}_B - \mathbf{r}_A)} = \sqrt{\{\mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)\}\cdot\{\mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)\}} = \sqrt{(\mathbf{p}_B - \mathbf{p}_A)\cdot\left\{\mathbf{R}^T\mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)\right\}}$$

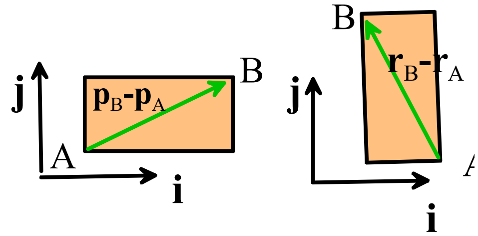
But we want the length of $\mathbf{r}_B - \mathbf{r}_A$ to equal the length of $\mathbf{p}_B - \mathbf{p}_A$, which means we need **R** to satisfy

$$\sqrt{(\mathbf{p}_B-\mathbf{p}_A)\cdot\left\{\mathbf{R}^T\mathbf{R}(\mathbf{p}_B-\mathbf{p}_A)\right\}}=\sqrt{(\mathbf{p}_B-\mathbf{p}_A)\cdot(\mathbf{p}_B-\mathbf{p}_A)}$$
$$\Rightarrow(\mathbf{p}_B-\mathbf{p}_A)\cdot\left\{\mathbf{R}^T\mathbf{R}(\mathbf{p}_B-\mathbf{p}_A)\right\}-(\mathbf{p}_B-\mathbf{p}_A)\cdot\{\mathbf{1}(\mathbf{p}_B-\mathbf{p}_A)\}=0$$
$$\Rightarrow(\mathbf{p}_B-\mathbf{p}_A)\cdot\left\{\left(\mathbf{R}^T\mathbf{R}-\mathbf{1}\right)(\mathbf{p}_B-\mathbf{p}_A)\right\}=0$$

where $\mathbf{1}$ is the identity tensor (we normally use **I** for the identity tensor, but rigid body dynamics uses **I** to denote the mass moment of inertia so it's already been taken….). With a bit of busy work, we can show that the last line can only be satisfied if $\mathbf{R}^T\mathbf{R}=\mathbf{1}$. In fact, a rigorous mathematical derivation of rotations *starts* with the statement that **R** must preserve the length of all vectors, and then derives all the other material in this section from that statement. This is not easy to follow the first time around, but will probably be the approach used in more advanced courses.

Examples:

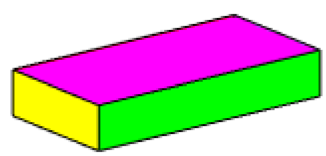
1. Write down the rotation matrix for the 2D rotation shown in the figure



The object rotates 90 degree counterclockwise about the **k** axis, so

$$\mathbf{R}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

2. The object shown in the figure is first rotated 90 degrees about the **i** axis, and then 180 degrees about the **j** axis. Find the rotation tensor.



We can construct the two rotations using the Rodriguez formula. For the first rotation $\theta=\pi/2$
$\mathbf{n}=\mathbf{i}\Rightarrow n_x=1 \quad n_y=n_z=0$

$$\mathbf{R}^{(1)}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}$$

For the second rotation $\theta=\pi$ $\mathbf{n}=\mathbf{j}\Rightarrow n_y=1 \quad n_x=n_z=0$

$$\mathbf{R}^{(2)}=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$$

The total rotation is therefore

$$\mathbf{R}=\mathbf{R}^{(2)}\mathbf{R}^{(1)}=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}-1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}$$

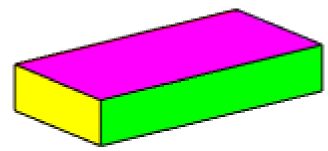
3. Find the axis-angle representation for the combined rotation in problem (2).

We can calculate the axis and angle of this rotation using the formulas

$$1+2\cos\theta=R_{xx}+R_{yy}+R_{zz}\Rightarrow 2\cos\theta=-2\Rightarrow\theta=\pi$$

$$\mathbf{n}=\sqrt{\frac{R_{xx}-\cos\theta}{1-\cos\theta}}\mathbf{i}\pm\sqrt{\frac{R_{yy}-\cos\theta}{1-\cos\theta}}\mathbf{j}\pm\sqrt{\frac{R_{zz}-\cos\theta}{1-\cos\theta}}\mathbf{k}$$

$$=\sqrt{\frac{-1-(-1)}{1-(-1)}}\mathbf{i}\pm\sqrt{\frac{0-(-1)}{1-(-1)}}\mathbf{j}\pm\sqrt{\frac{0-(-1)}{1-(-1)}}\mathbf{k}=\frac{1}{\sqrt{2}}(\mathbf{j}\pm\mathbf{k})$$



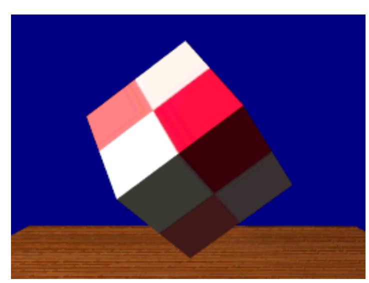
To decide which of these two choices to use we notice that $R_{yz}=-1$, which tells us that $n_y n_z<0$. The answer is therefore

$$\theta=\pi,\quad \mathbf{n}=\frac{1}{\sqrt{2}}(\mathbf{j}-\mathbf{k})$$

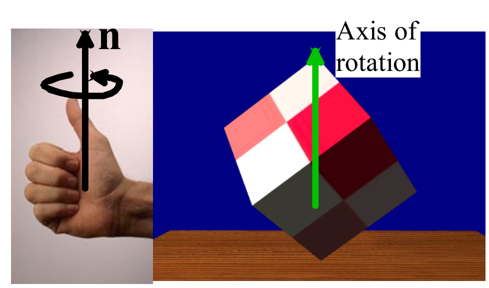
It is incredibly difficult to visualize the effect of a rotation about an arbitrary axis (at least for me). In fact this formula looks wrong – how can a 180 degree rotation end up tipping the box on its side? But the answer is right, as the animation (which will only show up in the html version of the notes) shows.

6.2.2 Describing rotational motion: The angular velocity vector and spin tensor

We described the location of a particle in space using its position vector, and its motion using velocity. We need to come up with something similar to velocity for rotations.



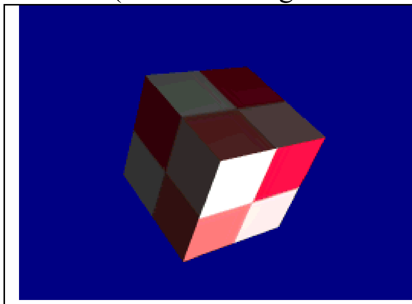
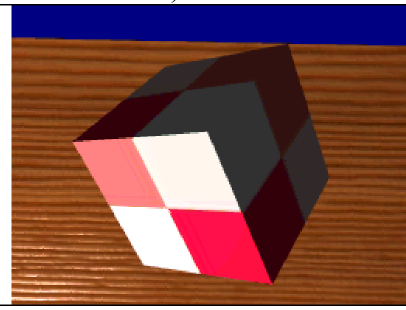
Definition of an angular velocity vector Visualize a spinning object, like the cube shown in the figure. The box rotates about an *axis* – in the example, the axis is the line connecting two cube diagonals. In addition, the object turns through some number of revolutions every minute. We would specify the angular velocity of the shaft as a vector $\boldsymbol{\omega}$, with the following properties:



1. The direction of the vector is parallel to the axis of the shaft (the axis of rotation). This direction would be specified by a unit vector **n** parallel to the shaft.

2. There are, of course, two possible directions for **n**. By convention, we always choose a direction such that, when viewed in a direction parallel to **n** (so the vector points away from you) the shaft appears

to rotate clockwise. Or conversely, if **n** points towards you, the shaft appears to rotate counterclockwise. (This is the `right hand screw convention')

	
Viewed along **n**	Viewed in direction opposite to **n**

3. The magnitude of the vector is the angular speed $d\theta / dt$ of the object, in radians per second. If you know the revs per minute n turned by the shaft, the number of radians per sec follows as $d\theta / dt = 120\pi n$. The magnitude of the angular velocity is often denoted by $\omega = d\theta / dt$

The angular velocity vector is then $\boldsymbol{\omega} = \dfrac{d\theta}{dt}\mathbf{n} = \omega\mathbf{n}$.

Since angular velocity is a vector, it has components $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$ in a fixed Cartesian basis.

As always, in two dimensions, everything is very simple. In this case objects can only rotate about the **k** axis, and we can write the angular velocity vector as

$$\boldsymbol{\omega} = \frac{d\theta}{dt}\mathbf{k}$$

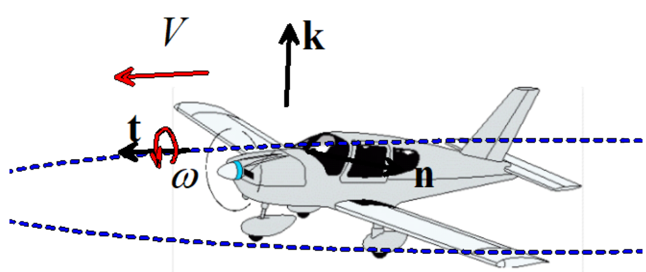
where θ is the counterclockwise angle of rotation of any line embedded in the body.

Writing down angular velocities:

For 2D problems, we always know the direction of the angular velocity and can just use $\boldsymbol{\omega} = \omega_z\mathbf{k}$ to write it down (of course if we know the value or a formula for ω_z we can use it).

For 3D problems, we can often use vector addition to write down $\boldsymbol{\omega}$. We can illustrate this with a simple example:

Example: The propeller on the aircraft shown in the figure spins (about its axis) at 2000 rpm. The aircraft travels at speed 200 km/hr in a turn with radius 1 km. What is the angular velocity vector of (i) the body of the aircraft, and (ii) the propeller? Express your answer in the normal-tangential-vertical basis.



(i) The circumference of the circle is $s = 2\pi R = 2\pi \text{km}$. The airplane completes a full circle in $t = s / V = (2\pi / 200) \times 3600 = 36\pi \text{sec}$. A full turn is 2π radians, so the aircraft body turns at a rate $2\pi / (36\pi) = (1/18)\mathbf{k}$ rad/s about the **k** axis.

(ii) The propeller turns at 2000 rpm *relative to the body of the plane*. The angular velocity of the prop with respect to a stationary observer is therefore the vector sum of the 2000 rpm about the **t** axis, plus the angular velocity of the body. This gives

$$\boldsymbol{\omega}_{prop} = \left[2000\times 2\pi/3600\right]\mathbf{t} + \frac{1}{18}\mathbf{k} = \frac{10\pi}{9}\mathbf{t} + \frac{1}{18}\mathbf{k} \quad \text{rad/s}$$

Relation between the rotation matrix and the angular velocity vector: the spin tensor

We might guess that the angular velocity vector is the derivative of the rotation tensor. This is sort of correct, but the full story is a bit more complicated. The relationship between **R** and $\boldsymbol{\omega}$ is constructed as follows:

1. We define the *spin tensor* **W** as

$$\mathbf{W} = \frac{d\mathbf{R}}{dt}\mathbf{R}^T$$

2. The spin tensor is always skew ($\mathbf{W} = -\mathbf{W}^T$), and we can read off the angular velocity vector by looking at its components. Specifically, if $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$ then

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

We can use this formula in two ways: (1) Given **R**, we can calculate **W** and then read off the angular velocity vector components. Alternatively, if we know $\boldsymbol{\omega}$, we can calculate **R** by first constructing **W**, then integrating the formula

$$\frac{d\mathbf{R}}{dt} = \mathbf{W}\mathbf{R}$$

Angular velocity-rotation relations in 2D

We can check this for the special case of a 2D rotation:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow \frac{d\mathbf{R}}{dt}\mathbf{R}^T = \begin{bmatrix} -\frac{d\theta}{dt}\sin\theta & -\frac{d\theta}{dt}\cos\theta \\ \frac{d\theta}{dt}\cos\theta & -\frac{d\theta}{dt}\sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 0 & -\frac{d\theta}{dt} \\ \frac{d\theta}{dt} & 0 \end{bmatrix}$$

As expected, we find that $\omega_z = \frac{d\theta}{dt}$.

This means that in 2D, angular velocity and the angle of rotation θ are related by the same formulas as distance traveled and speed for position. We can use all the same rules of calculus to go back and forth between them.

Angular velocity-Spin tensor formula

There is an important formula relating **W** and $\boldsymbol{\omega}$. Let $\mathbf{r}_B - \mathbf{r}_A$ be a vector joining any two points in a rigid body. Then

$$\mathbf{W}(\mathbf{r}_B - \mathbf{r}_A) = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)$$

You can see this by just multiplying out the definition of **W** and comparing the result to the cross product: if $\mathbf{r}_B - \mathbf{r}_A = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then

$$\mathbf{W}(\mathbf{r}_B - \mathbf{r}_A) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \omega_y z - \omega_z x \\ \omega_z x - \omega_x z \\ \omega_x y - \omega_y x \end{bmatrix}$$

Hopefully you can see that this is the same as the cross product!

6.2.3 The angular acceleration vector

Angular acceleration is the time derivative of angular velocity

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$$

For 3D, we can use

$$\alpha_x = \frac{d\omega_x}{dt} \qquad \alpha_y = \frac{d\omega_y}{dt} \qquad \alpha_z = \frac{d\omega_z}{dt}$$

For 3D, we can't express the angular accelerations or velocities as derivatives of rotation angles, because these can't be defined for a general motion.

For a 2D problem, the direction of angular velocity and acceleration are known, so we have

$$\boldsymbol{\alpha} = \alpha_z \mathbf{k} \qquad \boldsymbol{\omega} = \omega_z \mathbf{k}$$

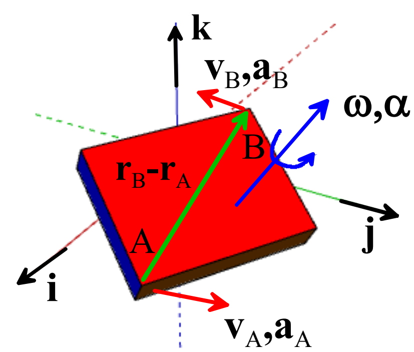
The components are related by

$$\alpha_z = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2} = \omega_z \frac{d\omega_z}{d\theta}$$

For 2D problems, we can use all the usual rules of calculus to go from angular acceleration to angular velocity to angle, and vice-versa (just like distance-speed-acceleration formulas for straight line motion).

6.2.3 Relative velocity and acceleration of two points in a rigid body

We now know how to describe rotational motion. Our next order of business is to discuss a couple of very important formulas that we use to analyze the motion of a system of rigid bodies, and also to derive formulas for the angular momentum and kinetic energy of a rigid body..



Consider a rigid body:

Let $\boldsymbol{\omega}$ be the (instantaneous) angular velocity of the body, and **W** the corresponding spin tensor

Let A and B be two arbitrary points in a rigid body, and let $\mathbf{r}_A, \mathbf{r}_B$ and $\mathbf{v}_A, \mathbf{v}_B$, $\mathbf{a}_A, \mathbf{a}_B$ be their (instantaneous) position, velocity and acceleration vectors.

Then the relative position and velocity of A and B are related by

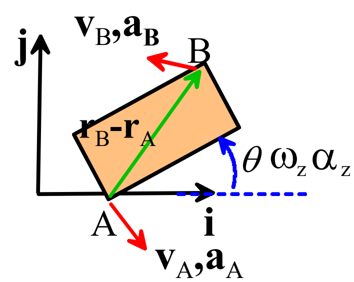
$$\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)$$
$$\mathbf{v}_B - \mathbf{v}_A = \mathbf{W}(\mathbf{r}_B - \mathbf{r}_A)$$

The relative acceleration of A and B are related their relative positions and velocity by

$$\mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times (\mathbf{v}_B - \mathbf{v}_A) = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A) \right]$$

For **2D problems only:** we can simplify these, because we know $\boldsymbol{\omega}$ is always parallel to the **k** direction. Therefore

$$\mathbf{v}_B - \mathbf{v}_A = \omega_z \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A)$$
$$\mathbf{a}_B - \mathbf{a}_A = \alpha_z \mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A) - \omega_z^2 (\mathbf{r}_B - \mathbf{r}_A)$$



Proof: These fornulas are easy to prove. Remember the mapping:

$$\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A) \quad \Rightarrow \quad \mathbf{v}_B - \mathbf{v}_A = \frac{d}{dt}(\mathbf{r}_B - \mathbf{r}_A) = \frac{d\mathbf{R}}{dt}(\mathbf{p}_B - \mathbf{p}_A)$$

Also,

$$\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A) \Rightarrow \mathbf{R}^T (\mathbf{r}_B - \mathbf{r}_A) = \mathbf{R}^T \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A) = (\mathbf{p}_B - \mathbf{p}_A)$$

Hence

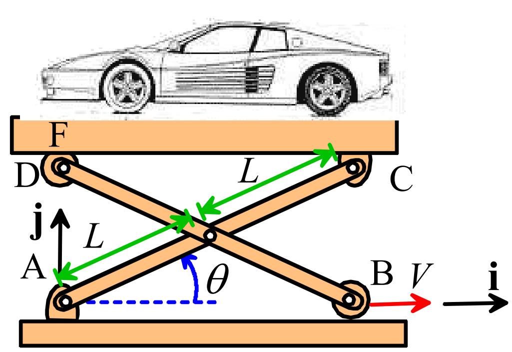
$$\mathbf{v}_B - \mathbf{v}_A = \frac{d\mathbf{R}}{dt}\mathbf{R}^T (\mathbf{r}_B - \mathbf{r}_A) = \mathbf{W}(\mathbf{r}_B - \mathbf{r}_A)$$

Remember that $\mathbf{W}(\mathbf{r}_B - \mathbf{r}_A) = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)$, so the acceleration formula then follows as

$$\mathbf{a}_B - \mathbf{a}_A = \frac{d}{dt}\left(\mathbf{v}_B - \mathbf{v}_A\right) = \frac{d\boldsymbol{\omega}}{dt} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times \frac{d}{dt}(\mathbf{r}_B - \mathbf{r}_A) = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times (\mathbf{v}_B - \mathbf{v}_A)$$

6.3 Analyzing motion in connected rigid bodies

The formulas in 6.2.3 are used to analyze motion in machines. A typical problem is illustrated in the figure. An actuator moves point B on the car jack shown in the figure horizontally with constant velocity V. What are the velocity and acceleration of the platform (CF)?



You could probably solve this rather simple example with elementary trig, but we need a more systematic method for general problems, especially to analyze 3D motion. Here's the general procedure

1. Define variables to denote the unknown angular

velocities and angular accelerations of each rigid body in the system

2. Write down all the known velocities in the system
3. Use the rigid body formulas

$$\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)$$

to write down equations relating velocities of the connections, joints, or contacts on each rigid body

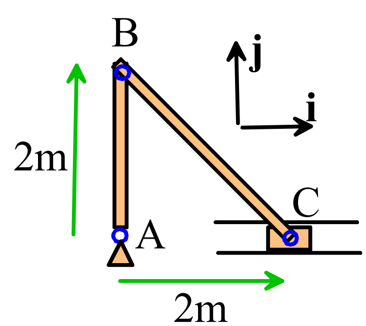
4. Write down *constraint equations* relating velocities of the two connected rigid bodies at each connection, joint, or contact
5. Solve the equations for unknown velocities of connections, and the angular velocities of the rigid bodies.
6. Finally, once the velocities are known, write down equations for the accelerations of pairs of joints/contacts/connections on each rigid body

$$\mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times (\mathbf{v}_B - \mathbf{v}_A)$$

7. Write down constraints equations for accelerations at connected points
8. Solve the equations in 6,7 for unknown accelerations and angular accelerations.

This all sounds terribly complicated, so let's solve a few examples to show how it works in practice.

Example 1: In the figure shown the link AB rotates counter-clockwise with constant angular speed 4 rad/s. Point C on member BC is constrained to move horizontally. Calculate the velocity and acceleration of point C.



Calculating the velocity:

- We know A is stationary, and are given the angular velocity of AB. We can use the rigid body formula to find the velocity of B:

$$\mathbf{v}_B - \mathbf{v}_A = \omega_{zAB}\mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A) = 4\mathbf{k} \times 2\mathbf{j}$$
$$\Rightarrow \mathbf{v}_B = -8\mathbf{i}$$

- We don't know the angular velocity of BC, so we introduce ω_{zBC} as an unknown, and use the rigid body formula for member BC to write down an equation for the velocity of C

$$\mathbf{v}_C - \mathbf{v}_B = \omega_{zBC}\mathbf{k} \times (\mathbf{r}_C - \mathbf{r}_B) = \omega_{zBC}\mathbf{k} \times (2\mathbf{i} - 2\mathbf{j}) = 2\omega_{BC}\mathbf{i} + 2\omega_{BC}\mathbf{j}$$
$$\Rightarrow \mathbf{v}_C = -8\mathbf{i} + 2\omega_{BC}\mathbf{i} + 2\omega_{BC}\mathbf{j}$$

- We know that C can only move horizontally. This means that its **j** component of velocity must be zero. This shows that

$$\omega_{BC} = 0, \ \mathbf{v}_C = -8\mathbf{i}$$

Calculating the acceleration:

- We know A is stationary, and are given the angular velocity and angular acceleration of AB. We can use the rigid body formula to find the acceleration of B:

$$\mathbf{a}_B - \mathbf{a}_A = \alpha_{zAB}\mathbf{k} \times (\mathbf{r}_B - \mathbf{r}_A) - \omega_{zAB}^2(\mathbf{r}_B - \mathbf{r}_A) = -32\mathbf{j}$$
$$\Rightarrow \mathbf{a}_B = -32\mathbf{j}$$

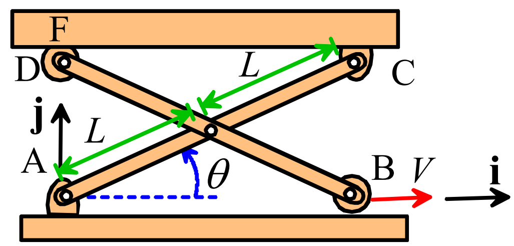
- We don't know the angular acceleration of BC, so we introduce α_{zBC} as an unknown and use the rigid body formula for member BC to write down an equation for the acceleration of C

$$\mathbf{a}_C - \mathbf{a}_B = \alpha_{zBC}\mathbf{k}\times(\mathbf{r}_C - \mathbf{r}_B) - \omega_{zAB}^2(\mathbf{r}_C - \mathbf{r}_B) = \alpha_{zBC}\mathbf{k}\times(2\mathbf{i}-2\mathbf{j}) - \mathbf{0}$$
$$\Rightarrow \mathbf{a}_C = -32\mathbf{j} + 2\alpha_{BC}\mathbf{i} + 2\alpha_{zBC}\mathbf{j}$$

- Point C can only move horizontally, so it can't have any vertical acceleration. This means that the **j** component of acceleration is zero:

$$2\alpha_{zBC} - 32 = 0 \Rightarrow \alpha_{zBC} = 16$$
$$\Rightarrow \mathbf{a}_C = 32\mathbf{i}$$

Example 2: For a more complicated example, we can solve the car jack problem posed at the start of this section. An actuator moves point B on the car jack shown in the figure horizontally with constant velocity V. What are the velocity and acceleration of the platform (CF)?



The system contains 3 rigid bodies (AC, BD, CF[3]). We don't know the angular velocities or accelerations of any of them, so we denote them by unknowns ω_{zAC}, ω_{zBD} ω_{zCF}, α_{zAC}, α_{zBD} α_{zCF}

Calculating the velocity:

- We start at point(s) with known velocity: A is stationary, and the velocity of B is given:

$$\mathbf{v}_A = \mathbf{0} \quad \mathbf{v}_B = V\mathbf{i}$$

- Point E lies on both member AC and on member BD. We use the rigid body formulas to write down an equation for the velocity of E on each member (notice we use the 2D equations):

$$\mathbf{v}_E - \mathbf{v}_A = \omega_{zAC}\mathbf{k}\times(\mathbf{r}_E - \mathbf{r}_A)$$
$$\mathbf{v}_E - \mathbf{v}_B = \omega_{zBD}\mathbf{k}\times(\mathbf{r}_E - \mathbf{r}_B)$$

- The two members AC and BD are pinned together at E and so must have the same velocity. We can eliminate $\mathbf{v}_E$ and write out the position vectors in **i**,**j** components

$$\omega_{zAC}\mathbf{k}\times(L\cos 30\mathbf{i} + L\sin 30\mathbf{j}) - V\mathbf{i} = \omega_{zBD}\mathbf{k}\times(-L\cos 30\mathbf{i} + L\sin 30\mathbf{j})$$
$$(-\omega_{zAC}L\sin 30 - V)\mathbf{i} + \omega_{zAC}L\cos 30\mathbf{j} = -\omega_{zBD}L\sin 30\mathbf{i} - \omega_{zBD}L\cos 30\mathbf{j}$$

The **i**,**j** components give two equations for ω_{zAC}, ω_{zBD}

$$-\omega_{zAC}L\sin\theta - V = -\omega_{zBD}L\sin\theta$$
$$\omega_{zAC}L\cos\theta = -\omega_{zBD}L\cos\theta$$
$$\Rightarrow -\omega_{zAC}2L\sin\theta\cos\theta - V\cos\theta = 0$$
$$\Rightarrow \omega_{zAC} = -V/(2L\sin\theta) \qquad \omega_{zBD} = V/(2L\sin\theta)$$

- We can now use the rigid body formulas for members AC and BD to find the velocities of C and D

[3] You may be wondering why only a single point was defined at C and E, but there are two points at D and F. That's because at C and E the members are pinned together, but there is a roller at D. At E, members AC, BD always have the same velocity and acceleration – we can just use a single variable to denote the velocity of this point. The same is true at C. Members CF and BD touch at F and D, but point D on AB does not have the same horizontal velocity as point F CF, so we need to be able to distinguish between them.

$$\mathbf{v}_C - \mathbf{v}_A = \omega_{zAC}\mathbf{k}\times(\mathbf{r}_C - \mathbf{r}_A) \Rightarrow \mathbf{v}_C = \frac{-V}{2L\sin\theta}\mathbf{k}\times(2L\cos\theta\mathbf{i} + 2L\sin\theta\mathbf{j}) = V\mathbf{i} - V\cot\theta\mathbf{j}$$

$$\mathbf{v}_D - \mathbf{v}_B = \omega_{zBD}\mathbf{k}\times(\mathbf{r}_D - \mathbf{r}_B) \Rightarrow \mathbf{v}_D = V\mathbf{i} + \frac{V}{2L\sin\theta}\mathbf{k}\times(2L\cos\theta\mathbf{i} + 2L\sin\theta\mathbf{j}) = -V\cot\theta\mathbf{j}$$

- We can use the rigid body formula for CF to relate the velocities of C and F
$$\mathbf{v}_F - \mathbf{v}_C = \omega_{zCF}\mathbf{k}\times(\mathbf{r}_F - \mathbf{r}_C)$$
$$\mathbf{v}_F = V\mathbf{i} - V\cot\theta\mathbf{j} - \omega_{zCF}2L\cos\theta\mathbf{j}$$
- Point D on CD and point F on CF must have the same vertical velocity (the roller at D allows their horizontal velocities to differ). This can be expressed as
$$\mathbf{v}_F\cdot\mathbf{j} = \mathbf{v}_D\cdot\mathbf{j} \Rightarrow -V\cot\theta - \omega_{zCF}2L\cos\theta = -V\cot\theta$$
$$\Rightarrow \omega_{zCF} = 0$$
- All points on CF therefore have the same velocity (equal to the velocity of C)
$$\mathbf{v}_{CF} = V\mathbf{i} - V\cot\theta\mathbf{j}$$

Calculating the acceleration.

- We can now calculate the accelerations. We start at a known point: Points A and B have zero acceleration.
- We can use the rigid body formula to calculate the acceleration of E on each of AC and BD:
$$\mathbf{a}_E - \mathbf{a}_A = \alpha_{zAC}\mathbf{k}\times(\mathbf{r}_E - \mathbf{r}_A) - \omega_{zAC}^2(\mathbf{r}_E - \mathbf{r}_A)$$
$$\mathbf{a}_E - \mathbf{a}_B = \alpha_{zAD}\mathbf{k}\times(\mathbf{r}_E - \mathbf{r}_B) - \omega_{zAD}^2(\mathbf{r}_E - \mathbf{r}_B)$$
- The two members are connected at E and so must have the same acceleration there. This shows that
$$\alpha_{zAC}\mathbf{k}\times(L\cos\theta\mathbf{i} + L\sin\theta\mathbf{j}) - \omega_{zAC}^2(L\cos\theta\mathbf{i} + L\sin\theta\mathbf{j})$$
$$= \alpha_{zAD}\mathbf{k}\times(-L\cos\theta\mathbf{i} + L\sin\theta\mathbf{j}) - \omega_{zAD}^2(-L\cos\theta\mathbf{i} + L\sin\theta\mathbf{j})$$
$$\Rightarrow \alpha_{zAC}(L\cos\theta\mathbf{j} - L\sin\theta\mathbf{i}) - \omega_{zAC}^2(L\cos\theta\mathbf{i} + L\sin\theta\mathbf{j})$$
$$= \alpha_{zAD}(-L\cos\theta\mathbf{j} - L\sin\theta\mathbf{i}) - \omega_{zAD}^2(-L\cos\theta\mathbf{i} + L\sin\theta\mathbf{j})$$
- The **i,j** components give two equations for the unknown angular accelerations:
$$-\alpha_{zAC}L\sin\theta - \omega_{zAC}^2L\cos\theta = -\alpha_{zAD}L\sin\theta + \omega_{zAD}^2L\cos\theta$$
$$\alpha_{zAC}L\cos\theta - \omega_{zAC}^2L\sin\theta = -\alpha_{zAD}L\cos\theta - \omega_{zAD}^2L\sin\theta$$
$$\Rightarrow -2\alpha_{zAC}L\sin\theta\cos\theta = \omega_{zAD}^2L(\cos^2\theta - \sin^2\theta) + \omega_{zAC}^2L \quad\Rightarrow\quad \alpha_{zAC} = -V^2\cos\theta/(4L^2\sin^3\theta)$$
$$\alpha_{zAD} = -\alpha_{zAC} = V^2\cos\theta/(4L^2\sin^3\theta)$$
- We can use the rigid body acceleration formulas to calculate the velocities of D and C:

$$\mathbf{a}_C - \mathbf{a}_A = \alpha_{zAC}\mathbf{k}\times(\mathbf{r}_C - \mathbf{r}_A) - \omega_{zAC}^2(\mathbf{r}_D - \mathbf{r}_A)$$

$$\mathbf{a}_C = -\frac{V^2\cos\theta}{4L^2\sin^3\theta}(2L\cos\theta\mathbf{j} - 2L\sin\theta\mathbf{i}) - \frac{V^2}{4L^2\sin^2\theta}(2L\cos\theta\mathbf{i} + 2L\sin\theta\mathbf{j})$$

$$\mathbf{a}_C = -\frac{V^2}{L}\frac{\cos\theta}{\sin^2\theta}\mathbf{i} - \frac{V^2}{2L}\frac{1}{\sin^3\theta}\mathbf{j}$$

$$\mathbf{a}_D - \mathbf{a}_B = \alpha_{zAD}\mathbf{k}\times(\mathbf{r}_D - \mathbf{r}_B) - \omega_{zAD}^2(\mathbf{r}_D - \mathbf{r}_B) \quad \Rightarrow \quad \mathbf{a}_D = -\frac{V^2}{2L}\frac{1}{\sin^3\theta}\mathbf{j}$$

- We can use the rigid body formula to relate the accelerations of C and F

$$\mathbf{a}_F - \mathbf{a}_C = \alpha_{zCF}\mathbf{k}\times(\mathbf{r}_F - \mathbf{r}_C) - \omega_{zCF}^2(\mathbf{r}_F - \mathbf{r}_C)$$

$$\Rightarrow \mathbf{a}_F = -\frac{V^2}{L}\frac{\cos\theta}{\sin^2\theta}\mathbf{i} - \frac{V^2}{2L}\frac{1}{\sin^3\theta}\mathbf{j} + \alpha_{zCF}\mathbf{k}\times(-2L\cos\theta\mathbf{i})$$

- Finally, we know that D and F must have the same vertical acceleration (so they remain in contact). Their horizontal accelerations may differ, because of the roller attached to D. This gives

$$\mathbf{a}_D\cdot\mathbf{j} = \mathbf{a}_F\cdot\mathbf{j}$$

$$\Rightarrow -\frac{V^2}{2L}\frac{1}{\sin^3\theta} - \alpha_{zCF}2L\cos\theta = -\frac{V^2}{2L}\frac{1}{\sin^3\theta} \Rightarrow \alpha_{CF} = 0$$

- Since CF has zero angular velocity and angular acceleration, all points on CF have the same acceleration (which must equal that of point C). Therefore

$$\mathbf{a}_{CF} = -\frac{V^2}{L}\frac{\cos\theta}{\sin^2\theta}\mathbf{i} - \frac{V^2}{2L}\frac{1}{\sin^3\theta}\mathbf{j}$$

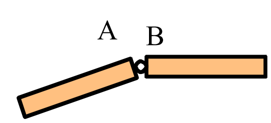
6.3.1 Summary of constraint equations at joints and contacts

As the examples in the preceding section show, the keys to analyzing motion in a system of connected rigid bodies are: (1) the formulas for relative velocity and acceleration of two points in a rigid body, and (2) *constraints* that relate the velocities and accelerations on two bodies at points where they touch.

There are three common types of connection between rigid bodies:

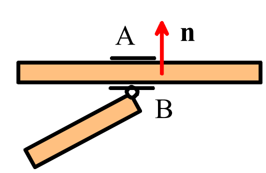
1. A pin joint: the two connected members must have the same velocity and acceleration at the connected point

$$\mathbf{v}_B = \mathbf{v}_A \qquad \mathbf{a}_B = \mathbf{a}_A$$



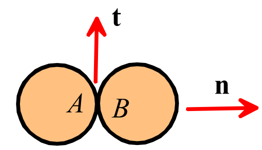
2. A slider joint: the two connected members must have the same velocity and acceleration normal to the slider

$$\mathbf{v}_B\cdot\mathbf{n} = \mathbf{v}_A\cdot\mathbf{n} \qquad \mathbf{a}_B\cdot\mathbf{n} = \mathbf{a}_A\cdot\mathbf{n}$$



3. Contact between two objects without relative slip (sliding) at the contact (friction forces must act to prevent the slip, in general): The velocities of the touching objects must be equal at the contact point. The tangential components of acceleration must also be equal (the normal components of acceleration differ)

$$\mathbf{v}_B = \mathbf{v}_A \qquad \mathbf{a}_B \cdot \mathbf{t} = \mathbf{a}_A \cdot \mathbf{t}$$



6.3.2 The Rolling Wheel

Wheels are everywhere. They can be analyzed using the general rigid body equations, but it's helpful to be able to avoid all the tedious cross products. In this section we summarize special formulas for velocity and acceleration of points on a wheel.

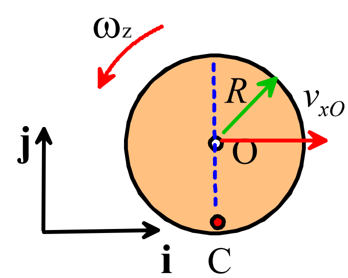
Motion of a wheel rolling without slip on a stationary surface

It is surprisingly difficult to visualize the motion of a wheel. The figure above might help: it shows the trajectory of one point on the circumference of the wheel. The point traces quite a complicated path. The important thing to notice is:

If a wheel rolls without slip on a stationary surface, the point touching the surface is stationary

Each point is only in contact with the ground for an instant, and while it touches the ground it has a large vertical acceleration, but it is instantaneously stationary. We know this from the list of constraints in Sect 6.3.1, of course, but it's still not an easy thing to visualize.

More generally, the ground need not necessarily be stationary (or the wheel could touch another surface). In this case we know that ***the contacting points on two bodies in rolling contact have equal velocity at the contact.***



Angular velocity-linear velocity formula: With this insight, we can use the rigid body formulas to calculate the instantaneous velocity vector for any point on the wheel. Assume that

- The wheel rolls with angular velocity $\boldsymbol{\omega} = \omega_z \mathbf{k}$ *counterclockwise rotation is positive.*
- The center of the wheel moves with velocity $\mathbf{v}_O = v_{xO}\mathbf{i}$

The rolling wheel formula gives

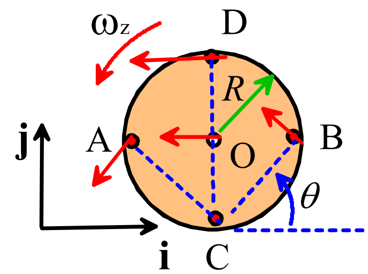
$$v_{xO} = -\omega_z R$$

To see this, you can simply use the rigid body formula to go from the contact point (which is stationary) to O

$$\mathbf{v}_O - \mathbf{v}_C = \boldsymbol{\omega} \times (\mathbf{r}_O - \mathbf{r}_C) \Rightarrow \mathbf{v}_O = \omega_z \mathbf{k} \times (R\mathbf{j}) = -\omega_z R\mathbf{i}$$

More generally, we can calculate the velocity of any point on the wheel we might be interested in. In fact, we can just write down the velocity of any point in the wheel by noticing that instantaneously all points are in circular motion about the contact point (just imagine the disk is rotating about C). See if you can show all the following:

- $\mathbf{v}_A = -\omega_z R(\mathbf{i} + \mathbf{j})$
- $\mathbf{v}_D = -\omega_z 2R\mathbf{i}$
- $\mathbf{v}_B = -\omega_z R(\mathbf{i} - \mathbf{j})$



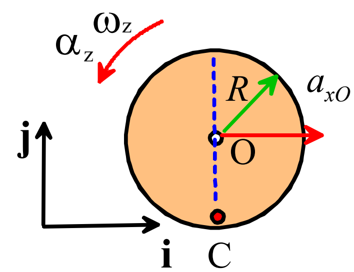
Notice that the direction of the velocity at each point is always perpendicular to the line connecting to the point to C.

Angular acceleration-linear acceleration formula: Assume that

- The wheel rolls with angular acceleration $\boldsymbol{\alpha} = \alpha_z \mathbf{k}$ *counterclockwise rotation is positive*.
- The center of the wheel moves with acceleration $\mathbf{a}_O = a_{xO}\mathbf{i}$

The rolling wheel formula gives

$$a_{xO} = -\alpha_z R$$



You can derive this formula in two different ways:

(1) Differentiate the velocity formula $v_{xO} = -\omega_z R$ with respect to time

(2) Use the rigid body formula:

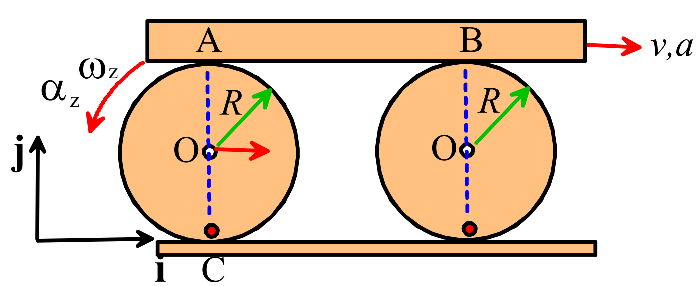
$$(\mathbf{a}_O - \mathbf{a}_C) = \boldsymbol{\alpha} \times (\mathbf{r}_O - \mathbf{r}_C) - \omega_z^2 (\mathbf{r}_O - \mathbf{r}_C)$$
$$\Rightarrow \mathbf{a}_O = \mathbf{a}_C - \alpha_z R\mathbf{i} - \omega_z^2 R\mathbf{j}$$

We know that the **i** component of acceleration at point C has to be the same as the **i** component of acceleration of the ground (i.e. zero). (The **j** components don't have to be equal). We also know that O has no **j** acceleration, because it remains at the same height above the ground. Therefore

$$a_{xO}\mathbf{i} = a_{yC}\mathbf{j} - \alpha_z R\mathbf{i} - \omega_z^2 R\mathbf{j}$$
$$\Rightarrow a_{xO} = -\alpha_z R \quad a_{yC} = \omega_z^2 R$$

We can calculate the acceleration of any other point on the disk using the rigid body formula.

Example: The block AB has horizontal acceleration a and horizontal speed v. Calculate the angular velocity and angular acceleration of the rollers. Then, calculate the linear velocity and acceleration of O



To solve problems like this we use two ideas: (1) the formulas relating velocity and accelerations of points on the disk; and (2) the tangential velocity and acceleration of contacting points are equal.

Here, we know the tangential velocity at C is zero; the tangential velocity at A is $v\mathbf{i}$. We can use the wheel formulas

$$v_{xA} = -2\omega_z R \Rightarrow \omega_z = -v/(2R)$$

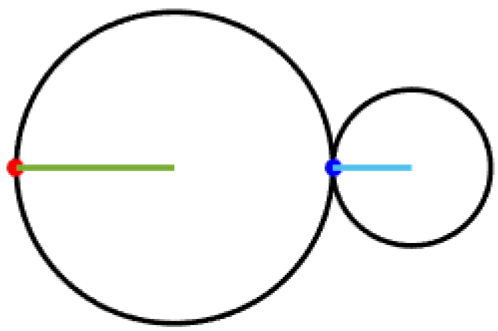
Similarly, the tangential acceleration at A is $a\mathbf{i}$. The rolling wheel formula gives

$$a_{xA} = -2\alpha_z R \Rightarrow \alpha_z = -a/(2R)$$

To find the velocity and acceleration at O, we can use

$$v_{xO} = -\omega_z R = v/2$$
$$a_{xO} = -\alpha_z R = a/2$$

6.3.3 Gears



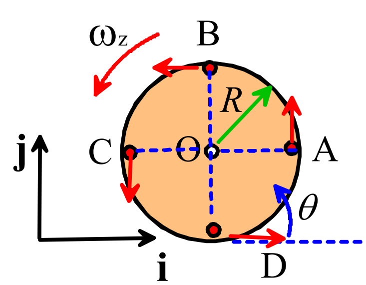
Gears can be analyzed in much the same way as a rolling wheel. Gears are used to increase or decrease angular velocities (they act like mechanical amplifiers): for example, in the animation the small gear is rotating at twice the angular rate of the large one. They also modify the *torques* (or moments) applied to the gears: if a gear system increases angular velocity, it reduces torque by the same factor (so the torque on the small gear in the animation is half that on the large one). Some clever gear systems can even be used to *add* angular velocities (see the discussion of epicyclic gears below.

There are many different gear designs. Here, we focus only on two-dimensional 'spur gears'. Spur gears have a rather complicated geometry, which we don't have time to discuss in detail in this course. They are designed to behave like two wheels which roll against each other with no slip at the contact. The wheel radius is equal to the 'pitch circle radius' of the gears (which is slightly smaller than physical diameter of the gears, because the teeth have to overlap). Gear manufacturers often specify the number of teeth on a gear instead of its radius. The number of teeth and the radius have to be related, because the teeth have to be the same circumferential distance apart for the gear pair to mesh.

We analyze motion of gears using two ideas:

(1) Two meshed gears must have equal velocities at the point where they touch.
(2) The rigid body formula, relating the velocity of points on the circumference of the gear to the velocity of its center:

$$\mathbf{v}_C = \mathbf{v}_O + \omega_z \mathbf{k} \times (\mathbf{r}_C - \mathbf{r}_O)$$

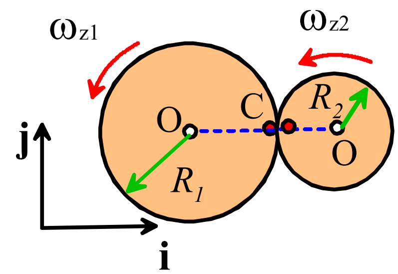


In practice we don't usually bother doing the cross product, and instead just write down the velocity on the circumference directly using the figure provided:

- $\mathbf{v}_A = \mathbf{v}_O + \omega_z R\mathbf{j}$
- $\mathbf{v}_B = \mathbf{v}_O - \omega_z R\mathbf{i}$
- $\mathbf{v}_C = \mathbf{v}_O - \omega_z R\mathbf{j}$
- $\mathbf{v}_D = \mathbf{v}_O + \omega_z R\mathbf{i}$

You don't have to remember these – just visualize every point on the gear moving in circular motion (counterclockwise) around O, and write down the vectors (be careful with signs!).

Example 1: The left gear in the figure rotates with counterclockwise angular velocity ω_{z1}. The large gear has radius R_1 and N_1 teeth, the small one has radius R_2 and N_2 teeth. Calculate the angular velocity of the smaller gear.



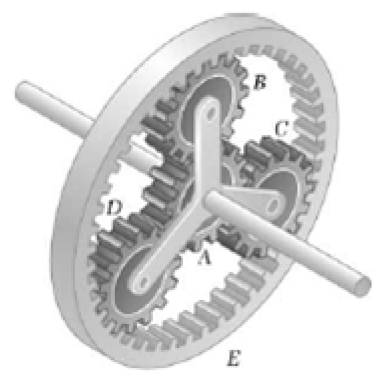
Note:

- The velocities of the two touching gears are equal at C
- The gear rotation/velocity formula gives

$$\omega_{z1} R_1 \mathbf{j} = -\omega_{z2} R_2 \mathbf{j} \Rightarrow \omega_{z2} = -\frac{R_2}{R_1}$$

Notice that we assume both gears rotate counterclockwise. The formula tells us that the second gear has a negative angular velocity – this means that it is actually rotating clockwise. The animation at the top of this section confirms that this indeed is the case.

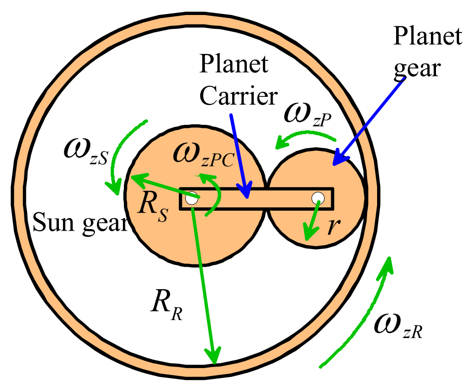
Example 2: An 'epicyclic' gearbox is a special arrangement of gears that has many applications. The sketch shows a simple example. The gearbox can be driven in three different places: one drive shaft is connected to the central sun gear (A); the other is attached to the 'planet carrier', which is joined to the center of the 'pinion gears' B,C and D. The outer gear (E – called the 'ring gear') can also be driven separately.



Epicyclic gearboxes are used in all automatic vehicle transmissions. They are also very useful in 'split power' drives, where two motors need to be connected together to drive a single axle. Hybrid vehicles, which have both an electric motor and an internal combustion engine driving the same axle, are one example. You can find a very nice description of the Toyota Prius split power transmission here: the website includes a Flash animation that lets you change the speeds of the motors in the system and visualize the motion of the gears.

The figure shows a schematic diagram illustrating the general geometry and motion of the system. We have four rigid bodies:

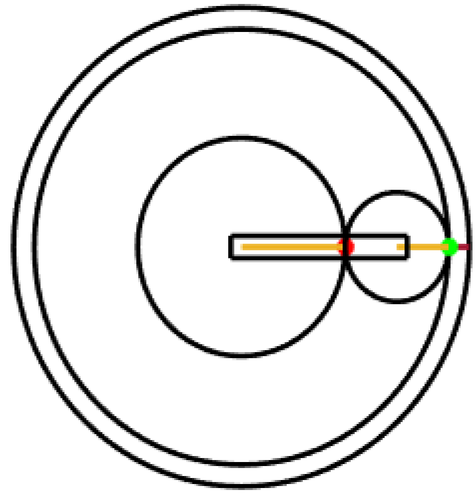
- The central sun gear, radius R_S , N_S teeth, rotating at angular velocity ω_{zS}
- The planet carrier, angular velocity ω_{zPC}
- The ring gear, radius R_R , with N_R teeth, angular velocity ω_{zR}
- The planet gear, radius $r = (R_R - R_S)/2$, $N_P = (N_R - N_S)/2$ teeth, rotating at angular velocity ω_{zP}



In any application, we are given the angular velocity of two of the drive shafts (any two of $\omega_{zS}, \omega_{zPC}, \omega_{zR}$), and must calculate the third. The planet gear is not connected to any drive shaft, so we usually don't care very much about its angular speed, but we will need to find ω_{zP} to solve for the unknown one of $\omega_{zS}, \omega_{zPC}, \omega_{zR}$.

This seems a terribly difficult problem, but it can be solved in a very simple way with a trick.

We start by solving a simpler version of the problem. Suppose that the planet carrier is stationary ($\omega_{zPC}=0$) and the sun gear rotates with angular speed ω_{zS} (see the animation). What is the angular velocity of the ring gear?



The sun gear and the planet gear are just a standard gear pair so we know that

$$\omega_{zS} R_S = -\omega_{zP} r \Rightarrow \omega_{zP} = -\omega_{zS} \frac{R_S}{r}$$

The two touching points on the planet gear and the ring gear must have the same velocity, so (using the rotating gear formula)

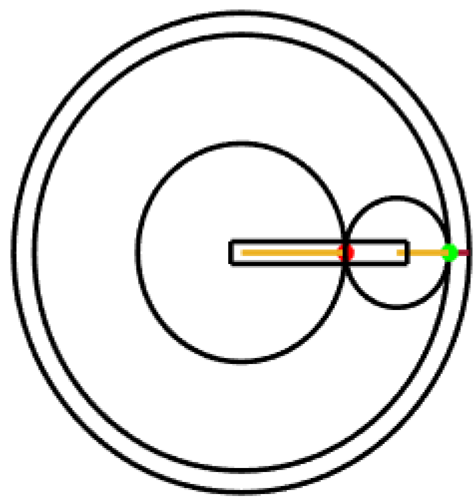
$$\omega_{zP} r \mathbf{j} = \omega_{zR} R_R \mathbf{j} \;\Rightarrow\; \omega_R = \omega_{zP} \frac{r}{R_R}$$

We can eliminate ω_{zP} to get the answer:

$$\omega_{zR} = -\omega_{zS} \frac{R_S}{R_R}$$

Now let's try the harder problem. The animation shows a general situation, where $\omega_{zS}, \omega_{zPC}$ are both nonzero. How can we find ω_{zR} now?

This is difficult to analyze because the center of the planet gear is not fixed, so it's hard for us to visualize the motion, and the standard gear formulas don't work. But we can simplify the problem by analyzing motion in a reference frame that rotates with the planet carrier. For example, imagine attaching a videocamera to the planet carrier – this camera would show the planet carrier to be stationary, with the surrounding world rotating in the opposite direction. *The angular velocity of the planet carrier would be subtracted from all the other angular velocities*. In this reference frame, we can use the result we just calculated:

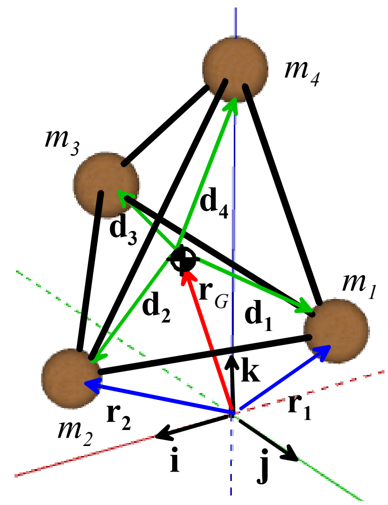


$$\frac{(\omega_{zR} - \omega_{zPC})}{(\omega_{zS} - \omega_{zPC})} = -\frac{R_S}{R_R}$$

This result is general, and can be re-arranged to tell you the angular velocities for any given combination of $\omega_{zS}, \omega_{zPC}$ and ω_{zR}.

6.4 Linear momentum, angular momentum and kinetic energy of rigid bodies

In this section, we determine how to calculate the angular momentum and kinetic energy of a rigid body, and define two important quantities: (1) the center of mass of a rigid body (which you already know), and (2) the Inertia tensor (matrix) of a rigid body.



To keep things simple, we won't consider a general rigid body right away. Instead, we will calculate the linear momentum, angular momentum, and kinetic energy of a system of N particles that are connected together by rigid, massless links.

Definitions of inertial properties: For this system, we will define

The total mass $M = \sum_{i=1}^{N} m_i$

The position of the center of mass $\mathbf{r}_G = \frac{1}{M}\sum_{i=1}^{N} m_i \mathbf{r}_i$

The position vector of each mass relative to the center of mass $\mathbf{d}_i = \mathbf{r}_i - \mathbf{r}_G$

The velocity of the center of mass $\mathbf{v}_G = \frac{d\mathbf{r}_G}{dt}$

The *mass moment of inertia about the center of mass* (a tensor, which can be expressed as a matrix if we choose a coordinate system and set $\mathbf{d}_i = d_{ix}\mathbf{i} + d_{iy}\mathbf{j} + d_{iz}\mathbf{k}$)

$$\mathbf{I}_G = \begin{bmatrix} I_{Gxx} & I_{Gxy} & I_{Gxz} \\ I_{Gyx} & I_{Gyy} & I_{Gyz} \\ I_{Gzx} & I_{Gzy} & I_{Gzz} \end{bmatrix} = \sum_{i=1}^{N} m_i \begin{bmatrix} d_{iy}^2 + d_{iz}^2 & -d_{ix}d_{iy} & -d_{ix}d_{iz} \\ -d_{ix}d_{iy} & d_{ix}^2 + d_{iz}^2 & -d_{iy}d_{iz} \\ -d_{ix}d_{iz} & -d_{iy}d_{iz} & d_{ix}^2 + d_{iy}^2 \end{bmatrix}$$

The mass moment of inertia is sometimes also written in a more abstract but very compact way as

$$\mathbf{I}_G = \sum_{i=1}^{N}\left(m_i \left|\mathbf{d}_i\right|^2 \mathbf{1} - m_i \mathbf{d}_i \otimes \mathbf{d}_i \right)$$

Here, $\mathbf{1}$ is the identity tensor, and $\mathbf{d}_i \otimes \mathbf{d}_i$ is a tensor with components $d_{ix}d_{ix}$, $d_{ix}d_{iy}$, $d_{ix}d_{iz}$, etc (the symbol $\otimes$ is called the 'diadic product' of two vectors).

Formulas for linear and angular momentum and kinetic energy: We will show that:

The total linear momentum is $\mathbf{p} = M\mathbf{v}_G$

The total angular momentum (about the origin) is $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G \boldsymbol{\omega}$

The total kinetic energy is $T = \frac{1}{2}M\mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_G \boldsymbol{\omega}$

These are actually general results that hold for all rigid bodies, as long as we use a more general definition of M and $\mathbf{I}_G$.

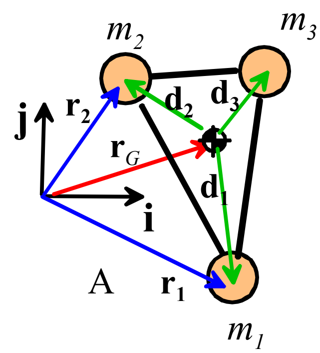
Simplified formulas for two dimensions: For planar problems, $d_{iz}=0$ (since all the masses are in the plane), and $\boldsymbol{\omega}=\omega_z\mathbf{k}$. In this case, we can use

The total linear momentum is $\mathbf{p}=M\mathbf{v}_G$

The total angular momentum (about the origin) is

$\mathbf{h}=\mathbf{r}_G\times M\mathbf{v}_G+I_{Gzz}\omega_z\mathbf{k}$

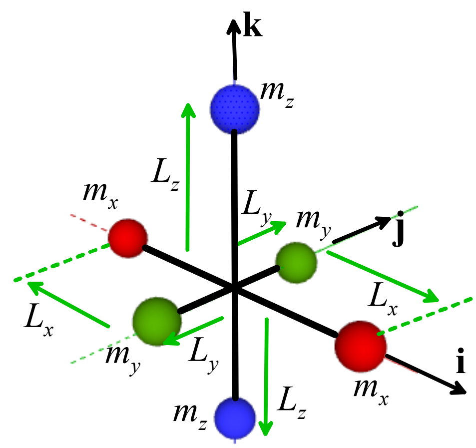
The total kinetic energy is $T=\frac{1}{2}M\mathbf{v}_G\cdot\mathbf{v}_G+\frac{1}{2}I_{Gzz}\omega_z^2$



Here I_{Gzz} is just the bottom diagonal term of the full inertia matrix (i.e. just a single number)

$$I_{Gzz}=\sum_{i=1}^{N}m_i\left(d_{ix}^2+d_{iy}^2\right)=\sum_{i=1}^{N}m_i\left|\mathbf{d}_i\right|^2$$

Example 1: A simple 3D assembly of masses is shown in the figure.



(1) Find the mass moment of inertia.

By symmetry, the COM is at the origin. The inertia tensor is therefore

$$\mathbf{I}_G=\begin{bmatrix}2\left(m_yL_y^2+m_zL_z^2\right) & 0 & 0\\ 0 & 2\left(m_xL_x^2+m_zL_z^2\right) & 0\\ 0 & 0 & 2\left(m_xL_x^2+m_yL_y^2\right)\end{bmatrix}$$

(2) Assume that the COM is stationary (i.e. the assembly rotates about the origin). Find formulas for the angular momentum and kinetic energy of the system, in terms of the angular velocity components $\omega_x,\omega_y,\omega_z$

The formula gives the angular momentum

$$\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega} = \begin{bmatrix} 2\left(m_y L_y^2 + m_z L_z^2\right) & 0 & 0 \\ 0 & 2\left(m_x L_x^2 + m_z L_z^2\right) & 0 \\ 0 & 0 & 2\left(m_x L_x^2 + m_y L_y^2\right) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= 2\left(m_y L_y^2 + m_z L_z^2\right)\omega_x \mathbf{i} + 2\left(m_x L_x^2 + m_z L_z^2\right)\omega_y \mathbf{j} + 2\left(m_x L_x^2 + m_y L_y^2\right)\omega_z \mathbf{k}$$

Note that **h** is a vector. Importantly, **h** is not generally parallel to the angular velocity vector, as this example shows.

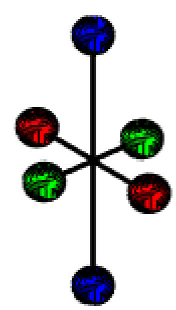
The kinetic energy is

$$T = \frac{1}{2}M\left|\mathbf{v}_G\right|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega} = \frac{1}{2}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \cdot \begin{bmatrix} 2\left(m_y L_y^2 + m_z L_z^2\right) & 0 & 0 \\ 0 & 2\left(m_x L_x^2 + m_z L_z^2\right) & 0 \\ 0 & 0 & 2\left(m_x L_x^2 + m_y L_y^2\right) \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$T = \left(m_y L_y^2 + m_z L_z^2\right)\omega_x^2 + \left(m_x L_x^2 + m_z L_z^2\right)\omega_y^2 + \left(m_x L_x^2 + m_y L_y^2\right)\omega_z^2$$

These results help us understand what the formulas are predicting. Note, for example, that:

- The mass moment of inertia always has the form *mass*length*^{2}. It has units of kg-m^2
- The mass moment of inertia is a measure of how mass is distributed about the center of mass. An object has a large inertia if the mass is far from the COM, and a small one if the mass is close to the COM.
- The matrix-vector products in the formulas for **h** and T are really just a way of calculating the velocity of each particle in the system in a quick way. For example, suppose we rotate our assembly of masses about the **k** axis with angular velocity ω_z (see the animation). Let's calculate the kinetic energy of the system, but without using the rigid body formulas. The two blue masses are stationary, so they have no KE. The red and green mass are both moving in a circle about the origin. The circular motion formula says their speed is $V = R\omega_z$ We can calculate the total kinetic energy using the usual formula



$$T = \sum_i \frac{1}{2} m_i V_i^2$$

$$= \frac{1}{2} 2m_x \left(L_x \omega_z\right)^2 + \frac{1}{2} 2m_y \left(L_y \omega_z\right)^2 = \left(m_x L_x^2 + m_y L_y^2\right)\omega_z^2$$

This explains why the formula for I_{Gzz} contains L_x and L_y - the I_{Gzz} component keeps track of how much energy or momentum is produced by a rotation about the z axis. The energy and momentum depend on the distances of the masses from the z axis – which of course depends on L_x and L_y.

Finally, note that we can interpret the two terms in the formulas for momentum and KE as quantifying (separately) the effects of translation and rotation

Angular momentum $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega}$
Translational + Rotational

Kinetic energy is $T = \frac{1}{2}M\mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_G\boldsymbol{\omega}$
Translational + Rotational

This helps explain why we can often idealize a system as a particle. If the rotational term is negligible, the angular momentum and kinetic energy of a rigid body is just the same as that of a particle located at the COM.

6.4.1 Deriving the linear momentum formula

By definition $\mathbf{p} = \sum_{i=1}^{N} m_i \mathbf{v}_i$. We can re-write this as follows:

$$\mathbf{p} = \sum_{i=1}^{N} m_i \mathbf{v}_i = \sum_{i=1}^{N} m_i \frac{d\mathbf{r}_i}{dt} = \frac{d}{dt}\sum_{i=1}^{N} m_i \mathbf{r}_i = \frac{d}{dt}\left(M\mathbf{r}_G\right) = M\mathbf{v}_G$$

(we used the definition of the COM to get the last result)

6.4.2 Deriving the angular momentum formula

Start with the definition: $\mathbf{h} = \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{v}_i$

Note that $\mathbf{r}_i = \mathbf{r}_G + \mathbf{d}_i$ and recall the relative velocity formula $\mathbf{v}_i - \mathbf{v}_G = \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_G) = \boldsymbol{\omega} \times \mathbf{d}_i$. This means we can re-write the angular momentum as

$$\begin{aligned}\mathbf{h} &= \sum_{i=1}^{N} (\mathbf{r}_G + \mathbf{d}_i) \times m_i (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{d}_i) \\ &= \left(\sum_{i=1}^{N} m_i\right)\mathbf{r}_G \times \mathbf{v}_G + \left(\sum_{i=1}^{N} m_i \mathbf{d}_i\right) \times \mathbf{v}_G + \mathbf{r}_G \times \boldsymbol{\omega} \times \left(\sum_{i=1}^{N} m_i \mathbf{d}_i\right) + \sum_{i=1}^{N} m_i \mathbf{d}_i \times \boldsymbol{\omega} \times \mathbf{d}_i\end{aligned}$$

Note that

$$\sum_{i=1}^{N} m_i \mathbf{d}_i = \sum_{i=1}^{N} m_i \left(\mathbf{r}_i - \mathbf{r}_G\right) = \sum_{i=1}^{N} m_i \mathbf{r}_i - \mathbf{r}_G \sum_{i=1}^{N} m_i = M\mathbf{r}_G - M\mathbf{r}_G = 0$$

Finally, recall the dreaded triple cross product formula

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

This means that

$$\mathbf{d}_i \times \boldsymbol{\omega} \times \mathbf{d}_i = (\mathbf{d}_i \cdot \mathbf{d}_i)\boldsymbol{\omega} - \mathbf{d}_i(\mathbf{d}_i \cdot \boldsymbol{\omega})$$

This gives us the result in compact notation directly

$$\sum_{i=1}^{N} m_i \mathbf{d}_i \times \boldsymbol{\omega} \times \mathbf{d}_i = \sum_{i=1}^{N} \left(m_i (\mathbf{d}_i \cdot \mathbf{d}_i)\boldsymbol{\omega} - m_i \mathbf{d}_i (\mathbf{d}_i \cdot \boldsymbol{\omega})\right) + \left[\sum_{i=1}^{N} \left(m_i \left|\mathbf{d}_i\right|^2 \mathbf{1} - m_i \mathbf{d}_i \otimes \mathbf{d}_i\right)\right] \cdot \boldsymbol{\omega} = \mathbf{I}_G \boldsymbol{\omega}$$

where we used the compact formula for the mass moment of inertia about the COM:

$$\mathbf{I}_G = \sum_{i=1}^{N} \left(m_i \left| \mathbf{d}_i \right|^2 \mathbf{1} - m_i \mathbf{d}_i \otimes \mathbf{d}_i \right)$$

If you don't like the compact formula, we can also get the matrix version by expand out the triple cross product

$$(\mathbf{d}_i \cdot \mathbf{d}_i)\boldsymbol{\omega} - \mathbf{d}_i(\mathbf{d}_i \cdot \boldsymbol{\omega}) = \begin{bmatrix} \omega_x(d_{ix}^2 + d_{iy}^2 + d_{iz}^2) \\ \omega_y(d_{ix}^2 + d_{iy}^2 + d_{iz}^2) \\ \omega_z(d_{ix}^2 + d_{iy}^2 + d_{iz}^2) \end{bmatrix} - \begin{bmatrix} d_{ix} \\ d_{iy} \\ d_{iz} \end{bmatrix} \left(\omega_x d_{ix} + \omega_y d_{iy} + \omega_z d_{iz} \right)$$

$$= \begin{bmatrix} d_{iy}^2 + d_{iz}^2 & -d_{ix}d_{iy} & -d_{ix}d_{iz} \\ -d_{ix}d_{iy} & d_{ix}^2 + d_{iz}^2 & -d_{iy}d_{iz} \\ -d_{ix}d_{iz} & -d_{iy}d_{iz} & d_{ix}^2 + d_{iy}^2 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

This again shows that

$$\sum_{i=1}^{N} m_i \mathbf{d}_i \times \boldsymbol{\omega} \times \mathbf{d}_i = \mathbf{I}_G \boldsymbol{\omega}$$

Finally collecting terms gives the required answer

$$\mathbf{h} = \left(\sum_{i=1}^{N} m_i \right) \mathbf{r}_G \times \mathbf{v}_G + \left(\sum_{i=1}^{N} m_i \mathbf{d}_i \right) \times \mathbf{v}_G + \sum_{i=1}^{N} m_i \mathbf{d}_i \times \boldsymbol{\omega} \times \mathbf{d}_i = \mathbf{r}_G \times M \mathbf{v}_G + \mathbf{I}_G \boldsymbol{\omega}$$

6.4.3 Deriving the kinetic energy formula

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i$$

We can use $\mathbf{v}_i - \mathbf{v}_G = \boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_G) = \boldsymbol{\omega} \times \mathbf{d}_i$

$$\sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \sum_{i=1}^{N} \frac{1}{2} m_i (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{d}_i) \cdot (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{d}_i)$$

$$= (\mathbf{v}_G \cdot \mathbf{v}_G) \sum_{i=1}^{N} \frac{1}{2} m_i + \mathbf{v}_G \cdot \boldsymbol{\omega} \times \left(\sum_{i=1}^{N} m_i \mathbf{d}_i \right) + \frac{1}{2} \sum_{i=1}^{N} m_i (\boldsymbol{\omega} \times \mathbf{d}_i) \cdot (\boldsymbol{\omega} \times \mathbf{d}_i)$$

Recall that

$$\left(\sum_{i=1}^{N} m_i \mathbf{d}_i \right) = \mathbf{0}$$

and expand the dot product of two cross products using the formula

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$$

This shows that

$$(\boldsymbol{\omega} \times \mathbf{d}_i) \cdot (\boldsymbol{\omega} \times \mathbf{d}_i) = \left(\boldsymbol{\omega} \cdot \boldsymbol{\omega} \right) \left(\mathbf{d}_i \cdot \mathbf{d}_i \right) - \left(\boldsymbol{\omega} \cdot \mathbf{d}_i \right)^2$$

As for the derivation of the angular momentum, this can be rearranged using the compact notation as

$$\sum_{i=1}^{N} m_i(\boldsymbol{\omega}\times\mathbf{d}_i)\cdot(\boldsymbol{\omega}\times\mathbf{d}_i) = \sum_{i=1}^{N}\left(m_i(\mathbf{d}_i\cdot\mathbf{d}_i)\boldsymbol{\omega}\cdot\boldsymbol{\omega} - m_i(\mathbf{d}_i\cdot\boldsymbol{\omega})^2\right) + \boldsymbol{\omega}\cdot\left[\sum_{i=1}^{N}\left(m_i\left|\mathbf{d}_i\right|^2\mathbf{1} - m_i\mathbf{d}_i\otimes\mathbf{d}_i\right)\right]\boldsymbol{\omega}$$
$$= \boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega}$$

Alternatively, we can get the matrix version of the formula as

$$(\boldsymbol{\omega}\times\mathbf{d}_i)\cdot(\boldsymbol{\omega}\times\mathbf{d}_i) = (\boldsymbol{\omega}\cdot\boldsymbol{\omega})(\mathbf{d}_i\cdot\mathbf{d}_i) - (\boldsymbol{\omega}\cdot\mathbf{d}_i)^2$$
$$= \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}\cdot\begin{bmatrix} d_{ix}^2+d_{iy}^2+d_{iz}^2 & 0 & 0\\ 0 & d_{ix}^2+d_{iy}^2+d_{iz}^2 & 0\\ 0 & 0 & d_{ix}^2+d_{iy}^2+d_{iz}^2\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix} - \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}\cdot\begin{bmatrix} d_{ix}^2 & d_{ix}d_{iy} & d_{ix}d_{iz}\\ d_{ix}d_{iy} & d_{iy}^2 & d_{iy}d_{iz}\\ d_{ix}d_{iz} & d_{iy}d_{iz} & d_{iz}^2\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$
$$= \begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}\cdot\begin{bmatrix} d_{iy}^2+d_{iz}^2 & -d_{ix}d_{iy} & -d_{ix}d_{iz}\\ -d_{ix}d_{iy} & d_{ix}^2+d_{iz}^2 & -d_{iy}d_{iz}\\ -d_{ix}d_{iz} & -d_{iy}d_{iz} & d_{ix}^2+d_{iy}^2\end{bmatrix}\begin{bmatrix}\omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$
$$= \boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega}$$

Finally, collecting all the terms gives the required answer

$$T = \sum_{i=1}^{N}\frac{1}{2}m_i\mathbf{v}_i\cdot\mathbf{v}_i = \frac{1}{2}M\mathbf{v}_G\cdot\mathbf{v}_G + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega}$$

6.4.4 Calculating the center of mass and inertia of a general rigid body

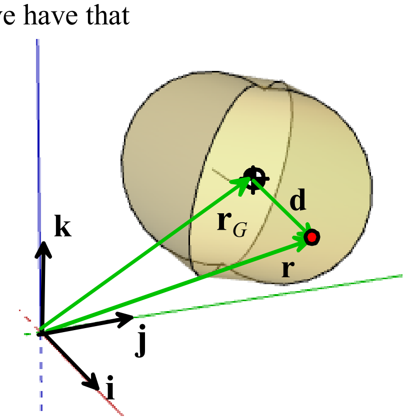
It is not hard to extend the results for a system of N particles to a general rigid body. We simply regard the body to be made up of an infinite number of vanishingly small particles, and take the limit of the sums as the particle volume goes to zero. The sums all turn into integrals.

3D problems: For a body with mass density ρ (mass per unit volume) we have that

- The total mass is $M = \int_V \rho dV$
- The position of the center of mass is $\mathbf{r}_G = \frac{1}{M}\int_V \mathbf{r}\rho dV$
- The mass moment of inertia about the center of mass is

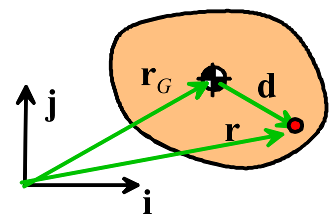
$$\mathbf{I}_G = \int_V \rho\begin{bmatrix} d_y^2+d_z^2 & -d_xd_y & -d_xd_z\\ -d_xd_y & d_x^2+d_z^2 & -d_yd_z\\ -d_xd_z & -d_yd_z & d_x^2+d_y^2\end{bmatrix}dV$$

where $\mathbf{d} = \mathbf{r} - \mathbf{r}_G$



For **2D problems:** We know the COM must lie in the **i,j** plane and we don't need to calculate the whole matrix.

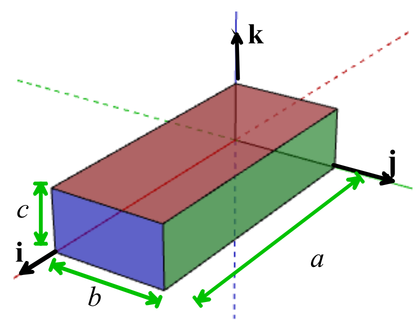
For a body with mass per unit area μ we can therefore use the formulas



- The total mass is $M = \int_A \mu dA$
- The position of the center of mass is $\mathbf{r}_G = \frac{1}{M}\int_A \mathbf{r}\mu dA$
- The mass moment of inertia about the center of mass is $I_{Gzz} = \frac{1}{M}\int_A \mu(d_x^2 + d_y^2)dA$

where $\mathbf{d} = \mathbf{r} - \mathbf{r}_G$

Example 1: To show how to use these, let's calculate the total mass, center of mass, and mass moment of inertia of a rectangular prism with faces perpendicular to the $\mathbf{i}, \mathbf{j}, \mathbf{k}$ axes:



First the total mass (sort of trivial)

$$M = \int_0^c \int_0^b \int_0^a \rho dxdydz = \rho abc$$

Now the COM

$$\mathbf{r}_G = \frac{1}{\rho abc}\int_0^c \int_0^b \int_0^a (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\rho dxdydz = \frac{1}{abc}\left(\frac{1}{2}a^2bc\mathbf{i} + a\frac{1}{2}b^2c\mathbf{j} + ab\frac{1}{2}c^2\mathbf{k}\right) = \frac{1}{2}(a\mathbf{i} + b\mathbf{j} + c\mathbf{k})$$

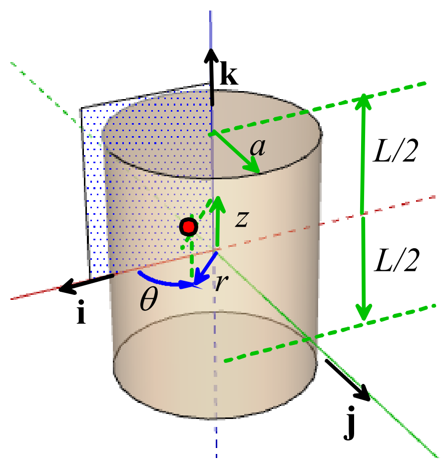
And finally the mass moment of inertia

$$\mathbf{I}_G = \int_0^c\int_0^b\int_0^a \begin{bmatrix} (y-b/2)^2+(z-c/2)^2 & -(x-a/2)(y-b/2) & -(x-a/2)(z-c/2) \\ sym & (x-a/2)^2+(z-c/2)^2 & -(y-b/2)(z-c/2) \\ sym & sym & (x-a/2)^2+(y-b/2)^2 \end{bmatrix} \rho dxdydz$$

$$= \rho \begin{bmatrix} \frac{1}{12}ab^3c+\frac{1}{12}abc^3 & 0 & 0 \\ 0 & \frac{1}{12}a^3bc+\frac{1}{12}abc^3 & 0 \\ 0 & 0 & \frac{1}{12}a^3bc+\frac{1}{12}ab^3c \end{bmatrix}$$

$$= \frac{M}{12}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

Example 2: As a second example, let's calculate the mass moment of inertia of a cylinder with mass density ρ , length L and radius a. We have to do the integrals with polar coordinates. For example, the inertia matrix is



$$\mathbf{I}_G^{Cylinder} = \int_{-L/2}^{L/2}\int_0^{2\pi}\int_0^a \begin{bmatrix} d_y^2+d_z^2 & -d_xd_y & -d_xd_z \\ -d_xd_y & d_x^2+d_z^2 & -d_yd_z \\ -d_xd_z & -d_yd_z & d_x^2+d_y^2 \end{bmatrix} \rho r dr d\theta dz$$

Now (in polar coordinates, and assuming that the COM is located at the center of the cylinder) $d_x = r\cos\theta \qquad d_y = r\sin\theta \qquad d_z = z$.

We can have Matlab do all the integrals for us:

```
clear all
syms x y z R r dx dy dz L theta a M rho mass real
dm = rho*R;
% Total mass
M = simplify(int(int(int(dm,R,[0,a]),theta,[0,2*pi]),z,[-L/2,L/2]))
r = [R*cos(theta),R*sin(theta),z];
% Position of COM
rG = simplify(int(int(int(r*dm,R,[0,a]),theta,[0,2*pi]),z,[-L/2,L/2])/M)
% Inertia matrix
dx = r(1)-rG(1); dy = r(2)-rG(2); dz = r(3)-rG(3);
integrand = dm*[dy^2+dz^2,-dx*dy,-dx*dz;...
                -dx*dy,dx^2+dz^2,-dy*dz;...
                -dx*dz,-dy*dz,dx^2+dy^2];
IG = simplify(int(int(int(integrand,R,[0,a]),theta,[0,2*pi]),z,[-L/2,L/2]))
% Rewrite the answer in terms of total mass instead of mass density
IGwithmass = simplify(mass*IG/M)
```

M = $\pi L a^2 \rho$

rG = $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$

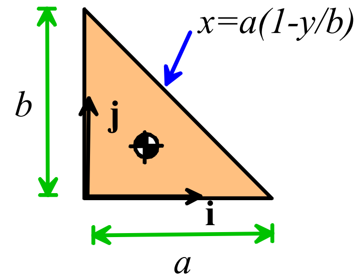
IG =

$$\begin{pmatrix} \frac{\pi L a^2 \rho \left(L^2 + 3a^2\right)}{12} & 0 & 0 \\ 0 & \frac{\pi L a^2 \rho \left(L^2 + 3a^2\right)}{12} & 0 \\ 0 & 0 & \frac{\pi L a^4 \rho}{2} \end{pmatrix}$$

IGwithmass =

$$\begin{pmatrix} \frac{\mathrm{mass}\left(L^2 + 3a^2\right)}{12} & 0 & 0 \\ 0 & \frac{\mathrm{mass}\left(L^2 + 3a^2\right)}{12} & 0 \\ 0 & 0 & \frac{a^2\,\mathrm{mass}}{2} \end{pmatrix}$$

Example 3: Let's finish up with a 2D example. Find the mass, center of mass, and out of plane mass moment of inertia of the triangle shown in the figure.



The total mass is $M = \int_0^b \int_0^{a(1-y/b)} \mu dxdy = \frac{1}{2}\mu ab$

The position of the COM is

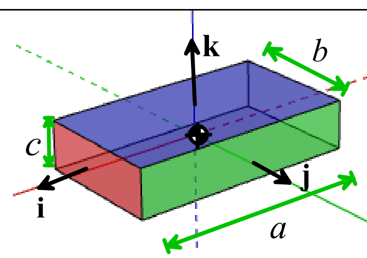
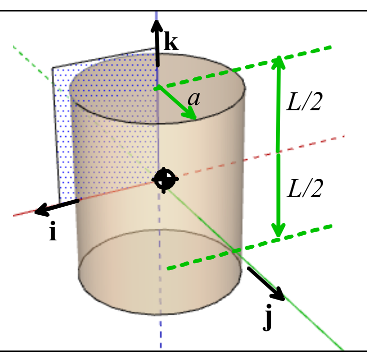
$$\mathbf{r}_G = \frac{2}{\mu ab}\int_0^b \int_0^{a(1-y/b)} (x\mathbf{i} + y\mathbf{j})\mu dxdy = \frac{1}{3}(a\mathbf{i} + b\mathbf{j})$$

The 2D mass moment of inertia is

$$I_{Gzz} = \int_0^b \int_0^{a(1-y/b)} \left((x - \frac{a}{3})^2 + (y - \frac{b}{3})^2 \right) \mu dxdy = \frac{ab\mu}{36}\left(a^2 + b^2\right) = \frac{M}{18}(a^2 + b^2)$$

This is all a big pain, and you may be contemplating a life of crime instead of an engineering career. Fortunately, it is very rare to have to do these sorts of integrals in practice, because all the integrals for common shapes have already been done. You can google most of them. The tables below give a short list of all the objects we will encounter in this course.

Table of mass moment of inertia tensors for selected 3D objects

Prism $M = \rho abc$		$\frac{M}{12}\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$
Solid Cylinder $M = \pi\rho a^2 L$		$\frac{ML^2}{12}\begin{bmatrix} 1 + 3a^2 / L^2 & 0 & 0 \\ 0 & 1 + 3a^2 / L^2 & 0 \\ 0 & 0 & 6a^2 / L^2 \end{bmatrix}$

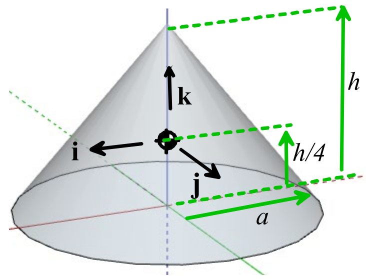
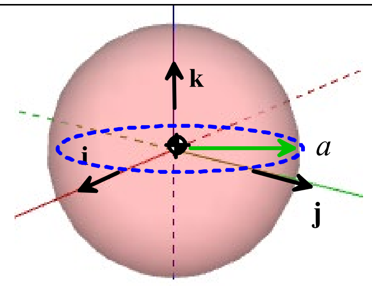
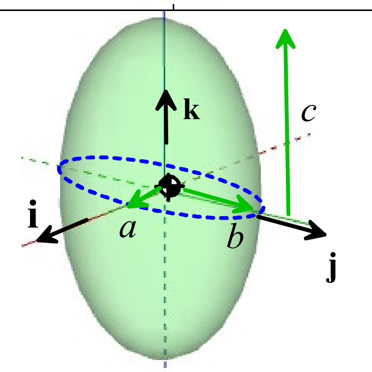
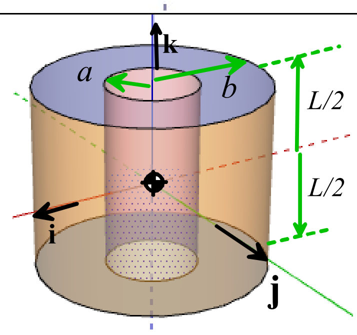
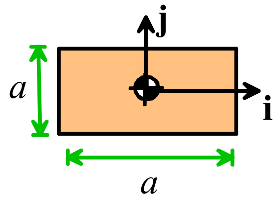
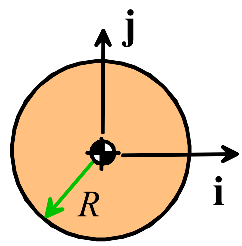
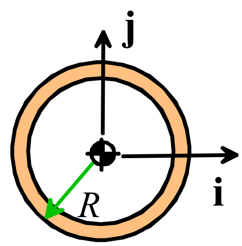
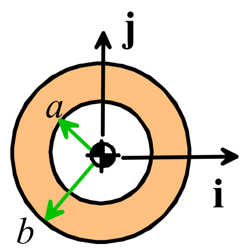
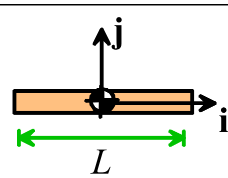
Solid Cone $M = \frac{\pi}{3}\rho a^2 h$		$\frac{3Ma^2}{20}\begin{bmatrix} 1+h^2/(4a^2) & 0 & 0 \\ 0 & 1+h^2/(4a^2) & 0 \\ 0 & 0 & 2 \end{bmatrix}$
Solid Sphere $M = \frac{4}{3}\pi\rho a^3$		$\frac{2Ma^2}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Solid Ellipsoid $M = \frac{4}{3}\pi\rho abc$		$\frac{M}{5}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$
Hollow Cylinder $M = \pi\rho(b^2-a^2)L$		$\frac{M}{12}\begin{bmatrix} L^2+3(a^2+b^2) & 0 & 0 \\ 0 & L^2+3(a^2+b^2) & 0 \\ 0 & 0 & 6(a^2+b^2) \end{bmatrix}$

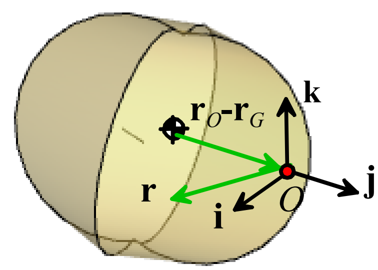
Table of mass moment of inertia about perpendicular axis for selected 2D objects

Square		$I_{Gzz} = \frac{M}{12}(a^2 + b^2)$
Disk		$I_{Gzz} = \frac{M}{2}R^2$
Thin ring		$I_{Gzz} = MR^2$
Hollow disk		$I_{Gzz} = \frac{M}{2}\left(a^2 + b^2\right)$
Slender rod		$I_{Gzz} = \frac{M}{12}L^2$

6.4.5 The Parallel Axis Theorem

In all the previous calculations we have been calculating the mass moment of inertia about the center of mass. This is what always appears in the general angular momentum formula. But we sometimes want to find the mass moment of inertia about a *different* point (not the COM). For example, if a body happens to be rotating about a fixed point, we can sometimes find its angular momentum and kinetic energy more quickly by first finding the mass moment of inertia about the fixed point, and then using special simpler formulas the angular momentum and kinetic energy (see section 6.4.10). We also sometimes want to find the combned mass moment of inertia of several bodies that are connected together. When we do this, we usually find the center of mass of the collection of bodies, and then add up the mass moments of inertia of all the separate bodies about the COM of the assembly (see section 6.4.6). To be able to do this, we need to be able to calculate the mass moment of inertia of a body about and arbitrary point, i.e. not the COM of the body.

The mass moment of inertia about an arbitrary point is defined exactly the same way as the inertia about the COM, except that we use the distances from our arbitrary point instead of the distance from the COM.



$$\mathbf{I}_O = \int_V \rho \begin{bmatrix} r_y^2 + r_z^2 & -r_x r_y & -r_x r_z \\ -r_x r_y & r_x^2 + r_z^2 & -r_y r_z \\ -r_x r_z & -r_y r_z & r_x^2 + r_y^2 \end{bmatrix} dV$$

It's painful to have to re-do all these integrals, however. If we already know $\mathbf{I}_G$, the parallel axis theorem lets us calculate $\mathbf{I}_O$ directly. Define the vector **d** that points from G to O

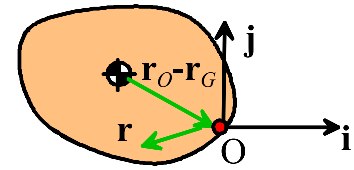
$$\mathbf{d} = \mathbf{r}_O - \mathbf{r}_G = d_x \mathbf{i} + d_y \mathbf{j} + d_z \mathbf{k}$$

Then for a 3D object with mass *M*

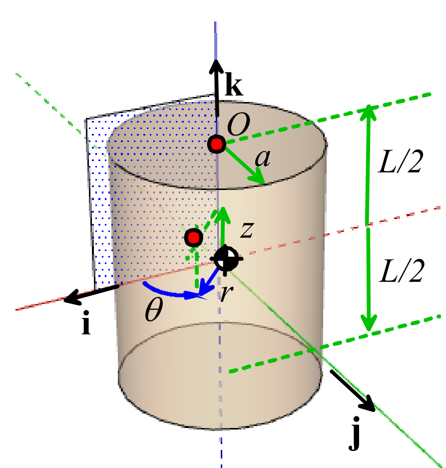
$$\mathbf{I}_O = \mathbf{I}_G + M \begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix}$$

For 2D we have a simpler result

$$I_{Ozz} = I_{Gzz} + M(d_x^2 + d_y^2)$$



Example: Let's find the mass moment of inertia of a cylinder about axes that pass through one end of the cylinder (O), instead of the COM.



Here, $\mathbf{d} = \frac{L}{2}\mathbf{k} \Rightarrow d_x = d_y = 0 \quad d_z = \frac{L}{2}$

The formula gives

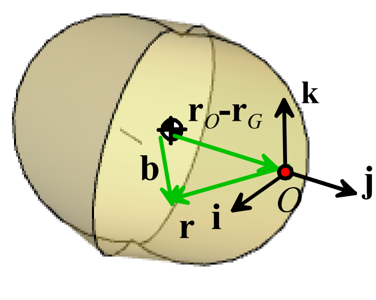
$$\mathbf{I}_O = \mathbf{I}_G + M\begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix}$$

$$= \frac{ML^2}{12}\begin{bmatrix} 1+3a^2/L^2 & 0 & 0 \\ 0 & 1+3a^2/L^2 & 0 \\ 0 & 0 & 6a^2/L^2 \end{bmatrix} + M\begin{bmatrix} L^2/4 & 0 & 0 \\ 0 & L^2/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \frac{ML^2}{12}\begin{bmatrix} 4+3a^2/L^2 & 0 & 0 \\ 0 & 4+3a^2/L^2 & 0 \\ 0 & 0 & 6a^2/L^2 \end{bmatrix}$$

Proof of the parallel axis theorem

Let $\mathbf{r}_O$ be some arbitrary point in space, and let $\mathbf{r}_G$ be the position of the COM. Define $\mathbf{d} = \mathbf{r}_O - \mathbf{r}_G$ as the vector from the COM to O, as shown in the figure.



Then let $\mathbf{r}$ denote the position vector of an infinitesimal volume element in the rigid body relative to O, and let $\mathbf{b}$ denote the position vector of the same volume element relative to the COM G. Then $\mathbf{r} = \mathbf{b} - \mathbf{d}$.

We also know that (by definition)

$$\mathbf{I}_O = \int_V \rho\left(|\mathbf{r}|^2 \mathbf{1} - \mathbf{r} \otimes \mathbf{r}\right) dV$$

$$\mathbf{I}_G = \int_V \rho\left(|\mathbf{b}|^2 \mathbf{1} - \mathbf{b} \otimes \mathbf{b}\right) dV$$

$$\int_V \rho \mathbf{b} dV = \mathbf{0} \qquad \int_V \rho dV = M$$

We can make use of $|\mathbf{r}|^2 = \mathbf{r} \cdot \mathbf{r}$ and then substitute $\mathbf{r} = \mathbf{b} - \mathbf{d}$ into (1). Expand the dot and dyadic product of $\mathbf{b} - \mathbf{d}$, note $\mathbf{d}$ is a constant and use the identities on the last line above, as follows

$$\mathbf{I}_O = \int_V \rho\left(|\mathbf{r}|^2\mathbf{1} - \mathbf{r}\otimes\mathbf{r}\right)dV = \int_V \rho\left((\mathbf{b}-\mathbf{d})\cdot(\mathbf{b}-\mathbf{d}) - (\mathbf{b}-\mathbf{d})\otimes(\mathbf{b}-\mathbf{d})\right)dV$$

$$= \int_V \rho\left((\mathbf{b}\cdot\mathbf{b})\mathbf{1} - \mathbf{b}\otimes\mathbf{b}\right)dV + \int_V \rho dV\left((\mathbf{d}\cdot\mathbf{d})\mathbf{1} - \mathbf{d}\otimes\mathbf{d}\right)$$

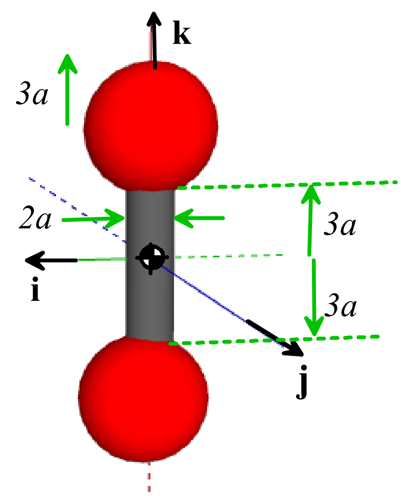
$$-\left(2\mathbf{d}\cdot\int_V \rho\mathbf{b}dV\right)\mathbf{1} + \left(\int_V \rho\mathbf{b}dV\right)\otimes\mathbf{d} + \mathbf{d}\otimes\left(\int_V \rho\mathbf{b}dV\right)$$

$$= \int_V \rho\left(|\mathbf{b}|^2\mathbf{1} - \mathbf{b}\otimes\mathbf{b}\right)dV + M\left(|\mathbf{d}|^2\mathbf{1} - \mathbf{d}\otimes\mathbf{d}\right) = \mathbf{I}_G + M\left(|\mathbf{d}|^2\mathbf{1} - \mathbf{d}\otimes\mathbf{d}\right)$$

6.4.6 Calculating moments of inertia of complex shapes by summation

The most important application of the parallel axis theorem is in calculating the mass moment of inertia of complicated objects (which don't appear in our table) by adding together moments of inertia for simple shapes. We can illustrate this with a couple of simple examples.

Example 1: Two spheres with radius $3a$ are connected by a rigid cylinder with length $6a$ and radius a to create a dumbbell. All objects have the same mass density ρ. Calculate the total mass moment of inertia of the dumbbell.



The general approach is

(1) Find the COM of the entire assembly
(2) Find the mass moment of inertia of each shape (the spheres and the cylinder) about its own COM
(3) Use the parallel axis theorem to find the moment of inertia of each shape about the combined COM
(4) Add all the moments of inertia

For our problem

(1) We know the COM is at the origin by symmetry, so we don't need to calculate it
(2) The inertia matrices of each object (cylinder + sphere) about their own COM are:

$$\mathbf{I}_G^{sphere} = \frac{2}{5}\left(\frac{4\pi}{3}(3a)^3\rho\right)\begin{bmatrix}(3a)^2 & 0 & 0\\ 0 & (3a)^2 & 0\\ 0 & 0 & (3a)^2\end{bmatrix}$$

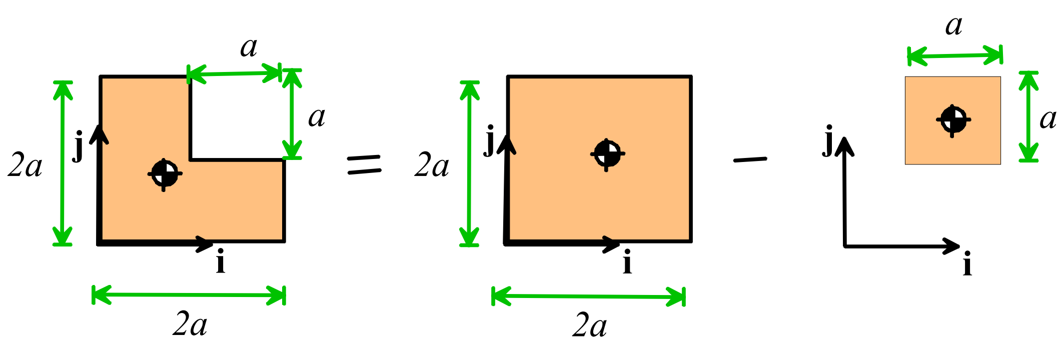
$$\mathbf{I}_G^{cylinder} = \frac{1}{12}\left(6\pi a^3\rho\right)\begin{bmatrix}(6a)^2+3a^2 & 0 & 0\\ 0 & (6a)^2+3a^2 & 0\\ 0 & 0 & 6a^2\end{bmatrix}$$

(3) We don't need to use the parallel axis theorem for the cylinder, because its COM is already at the same place as the COM of the assembly. For the spheres, we need to move the COM a distance *6a* parallel to the **k** direction. This means that $d_x = d_y = 0, \quad d_z = 6a$ in our formula. Therefore

$$\mathbf{I}_{COM}^{sphere} = \frac{2}{5}\left(\frac{4\pi}{3}(3a)^3\rho\right)\begin{bmatrix}(3a)^2 & 0 & 0\\ 0 & (3a)^2 & 0\\ 0 & 0 & (3a)^2\end{bmatrix} + \left(\frac{4\pi}{3}(3a)^3\rho\right)\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & (6a)^2\end{bmatrix}$$

(4) We can add everything up (note that there are two spheres). Its best to use Mupad. The answer is

$$\mathbf{I}_{COM} = \frac{1}{140}M\begin{bmatrix}929a^2 & 0 & 0\\ 0 & 929a^2 & 0\\ 0 & 0 & 9514a^2\end{bmatrix} \qquad M = 42\pi a^3\rho$$



Example 2: Things are a lot simpler in 2D. The procedure is the same, but we only need to calculate I_{zz}. For example, to calculate the mass moment of inertia for a square *2ax2a* plate with a hole with an *axa* square cut out from the top corner we would use the following approach.

Start by calculating the total mass and the position of the COM. We can regard the cut-out section as a square with negative density inside a larger 2ax2a square.

The total mass is therefore $M = \rho(2a)^2 - \rho a^2 = 3\rho a^2$

The position of the COM is $\mathbf{r}_G = \frac{1}{M}\left(4a^2\rho(a\mathbf{i} + a\mathbf{j}) - a^2\rho(\frac{3a}{2}\mathbf{i} + \frac{3a}{2}\mathbf{j})\right) = \frac{5}{6}a(\mathbf{i} + \mathbf{j})$

The mass moment of inertia of the 2ax2a square and the axa square are

Large square $I_{Gzz} = \frac{1}{12}4\rho a^2(4a^2 + 4a^2) = \frac{8}{3}\rho a^4$ (COM at $a(\mathbf{i} + \mathbf{j})$)

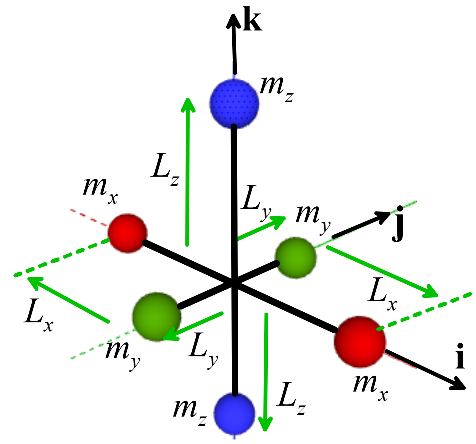
Small square $I_{Gzz} = -\frac{1}{12}\rho a^2(a^2 + a^2) = -\frac{1}{6}\rho a^4$ (COM at $\frac{3}{2}a(\mathbf{i} + \mathbf{j})$)

We now use the parallel axis theorem to find the moment of inertia of each square about the combined COM. For the large square: $d_x = \frac{1}{6}a \quad d_y = \frac{1}{6}a$. For the small square, $d_x = \frac{2}{3}a \quad d_y = \frac{2}{3}a$. The total mass moment of inertia is therefore

$$I_{Gzz}^{total} = \frac{8}{3}\rho a^4 + 4a^2\rho\left(\frac{1}{36}a^2 + \frac{1}{36}a^2\right) - \frac{1}{6}\rho a^4 - \rho a^2\left(\frac{4}{9}a^2 + \frac{4}{9}a^2\right) = \frac{11}{6}\rho a^4 = \frac{11}{18}Ma^2$$

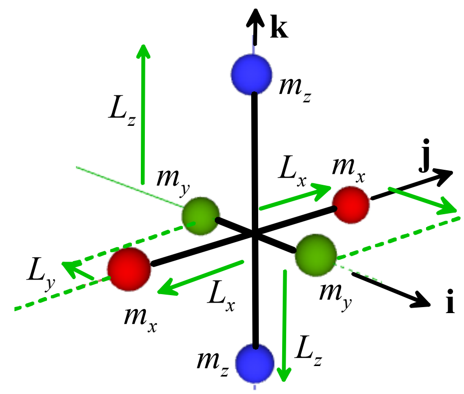
6.4.7 Rotating the inertia tensor

All the curious properties of spinning objects – a gyroscope; a boomerang; the rattleback – are consequences of the fact that ***the mass moment of inertia of an object changes when it is rotated***. We can see this very easily by re-visiting our assembly of masses. In the original calculation, the red, green and blue masses were located on the **i,j,k** axes. We calculated the inertia tensor to be



$$\mathbf{I}_G = \begin{bmatrix} 2\left(m_y L_y^2 + m_z L_z^2\right) & 0 & 0 \\ 0 & 2\left(m_x L_x^2 + m_z L_z^2\right) & 0 \\ 0 & 0 & 2\left(m_x L_x^2 + m_y L_y^2\right) \end{bmatrix}$$

Now suppose we rotate the assembly through 90 degrees about the **k** axis. The red masses now lie on the **j** axis, and the green ones line up with the **i** axis. It is not hard to see that the new mass moment of inertia is now



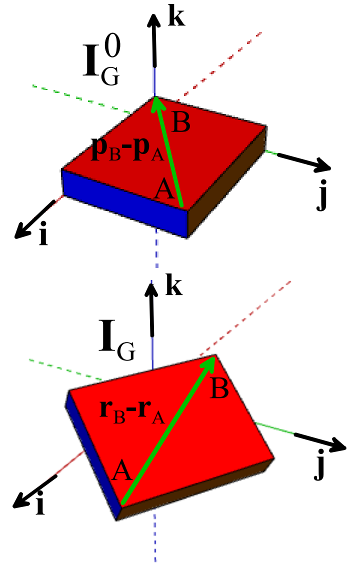
$$\mathbf{I}_G = \begin{bmatrix} 2\left(m_x L_x^2 + m_z L_z^2\right) & 0 & 0 \\ 0 & 2\left(m_y L_y^2 + m_z L_z^2\right) & 0 \\ 0 & 0 & 2\left(m_x L_x^2 + m_y L_y^2\right) \end{bmatrix}$$

(I_{xx}, I_{yy} have switched positions)

This seems like a huge problem – if we needed to re-calculate the mass moment of inertia from scratch every time a rigid body moves, analyzing rigid body motion would be nearly impossible.

Fortunately, we can derive a formula that tells us how the mass moment of inertia of a body changes when it is rotated.

Rotation formula for moments of inertia: Consider the rectangular prism shown in the figure. Let $\mathbf{I}_G^0$ denote the mass moment of inertia with the prism oriented so the faces are perpendicular to **i,j,k** (i.e. the inertia given in the table in Sect 6.4.5).



Suppose the body is then rotated by a tensor **R**.

The mass moment of inertia after rotation is given by

$$\mathbf{I}_G = \mathbf{R}\mathbf{I}_G^0\mathbf{R}^T$$

Example: The prism shown in the figure is rotated by 45 degrees about the **k** axis. Calculate the mass moment of inertia after the rotation

Start by calculating the rotation (use the formulas from 6.2.1)

$$\mathbf{R} = \begin{bmatrix} \cos(45) & -\sin(45) & 0 \\ \sin(45) & \cos(45) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We know the inertia tensor of the prism before it is rotated is

$$\mathbf{I}_G^0 = \frac{M}{12}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

We can use Matlab to do the tedious matrix multiplications

```
syms a b c real
IG0 = [b^2+c^2,0,0;...
        0,a^2+c^2,0;...
        0,0,a^2+b^2];
R = [cos(pi/4),-sin(pi/4),0;...
     sin(pi/4),cos(pi/4),0;...
     0,0,1];
 IG = simplify(R*IG0*transpose(R))
```

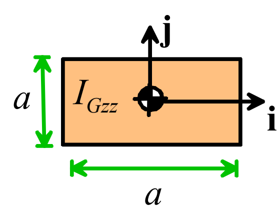
IG =

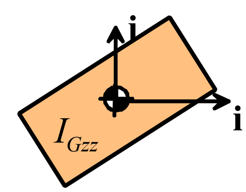
$$\begin{pmatrix} \frac{a^2}{2}+\frac{b^2}{2}+c^2 & \frac{b^2}{2}-\frac{a^2}{2} & 0 \\ \frac{b^2}{2}-\frac{a^2}{2} & \frac{a^2}{2}+\frac{b^2}{2}+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix}$$

Note that the inertia tensor is no longer diagonal.

Rotation formula for 2D motion: Fortunately, 2D is simple

Rotating a 2D object about the* k *axis does not change I_{Gzz}





Proof of the rotation formula: Consider a system of N particles. Suppose that before rotation, the particles are at positions $\mathbf{d}_i^0$ relative to the COM. The initial inertia tensor is

$$\mathbf{I}_G^0 = \sum_{i=1}^{N} \left(m_i \left| \mathbf{d}_i^0 \right|^2 \mathbf{1} - m_i \mathbf{d}_i^0 \otimes \mathbf{d}_i^0 \right)$$

Now rotate the system, so the particle s move to new positions $\mathbf{d}_i = \mathbf{R}\mathbf{d}_i^0$. The new inertia tensor is

$$\mathbf{I}_G = \sum_{i=1}^{N} \left(m_i \left| \mathbf{d}_i \right|^2 \mathbf{1} - m_i \mathbf{d}_i \otimes \mathbf{d}_i \right)$$

Recall that $\mathbf{R}\mathbf{R}^T = \mathbf{1}$ and recall that a rotation $\mathbf{R}$ does not change lengths so $\left| \mathbf{d}_i^0 \right| = \left| \mathbf{d}_i \right|$. Therefore

$$\mathbf{I}_G = \sum_{i=1}^{N} \left(m_i \left| \mathbf{d}_i^0 \right|^2 \mathbf{R}\mathbf{R}^T - m_i (\mathbf{R}\mathbf{d}_i^0) \otimes (\mathbf{R}\mathbf{d}_i^0) \right)$$

It is easy to show (just write out the matrix products) $(\mathbf{R}\mathbf{d}_i^0) \otimes (\mathbf{R}\mathbf{d}_i^0) = \mathbf{R}(\mathbf{d}_i^0 \otimes \mathbf{d}_i^0)\mathbf{R}^T$, which shows that

$$\mathbf{I}_G = \sum_{i=1}^{N} \left(m_i \left| \mathbf{d}_i^0 \right|^2 \mathbf{R}\mathbf{R}^T - m_i \mathbf{R}(\mathbf{d}_i^0 \otimes \mathbf{d}_i^0)\mathbf{R}^T \right) = \mathbf{R} \sum_{i=1}^{N} \left(m_i \left| \mathbf{d}_i^0 \right|^2 \mathbf{1} - m_i \mathbf{d}_i^0 \otimes \mathbf{d}_i^0 \right) \mathbf{R}^T = \mathbf{R}\mathbf{I}_G^0 \mathbf{R}^T$$

6.4.8 Time derivative of the inertia tensor

When we analyze motion of a rigid body, we will need to calculate the time derivatives of the linear and angular momentum. Linear momentum is no problem, but for angular momentum, we will need to know how to differentiate $\mathbf{I}_G$ with respect to time. There is a formula for this:

$$\frac{d\mathbf{I}_G}{dt} = \mathbf{W}\mathbf{I}_G - \mathbf{I}_G \mathbf{W}$$

where $\mathbf{W} = \dfrac{d\mathbf{R}}{dt}\mathbf{R}^T$ is the spin tensor (see sect 6.2.2)

Proof:

- Start with $\mathbf{I}_G = \mathbf{R}\mathbf{I}_G^0 \mathbf{R}^T$ and take the time derivative

$$\frac{d\mathbf{I}_G}{dt} = \frac{d\mathbf{R}}{dt}\mathbf{I}_G^0 \mathbf{R}^T + \mathbf{R}\mathbf{I}_G^0 \frac{d\mathbf{R}^T}{dt}$$

- Recall that $\mathbf{R}\mathbf{R}^T = \mathbf{1} \Rightarrow \dfrac{d\mathbf{R}}{dt}\mathbf{R}^T + \mathbf{R}\dfrac{d\mathbf{R}^T}{dt} = \mathbf{0} \Rightarrow \dfrac{d\mathbf{R}^T}{dt} = -\mathbf{R}^T \dfrac{d\mathbf{R}}{dt}\mathbf{R}^T = -\mathbf{R}^T \mathbf{W}$
- Finally note that $d\mathbf{R}/dt = \mathbf{W}\mathbf{R}$ and therefore

$$\frac{d\mathbf{I}_G}{dt} = \mathbf{W}\mathbf{R}\mathbf{I}_G^0 \mathbf{R}^T + \mathbf{R}\mathbf{I}_G^0 \mathbf{R}^T \mathbf{W} = \mathbf{W}\mathbf{I}_G - \mathbf{I}_G \mathbf{W}$$

6.4.9 Time derivative of angular momentum

To use the angular momentum conservation equation, we will need to know how to calculate the time derivative of the angular momentum. When we do this for a 3D problem, we need to take into account that the mass moment of inertia changes as the body rotates. We will prove the following formula:

$$\frac{d\mathbf{h}}{dt} = \mathbf{r}_G \times M\mathbf{a}_G + \mathbf{I}_G\boldsymbol{\alpha} + \boldsymbol{\omega} \times \left(\mathbf{I}_G\boldsymbol{\omega}\right)$$

For **2D planar problems** this can be simplified to:

$$\frac{d\mathbf{h}}{dt} = \mathbf{r}_G \times M\mathbf{a}_G + I_{Gzz}\alpha_z\mathbf{k}$$

Proof: We start by taking the time derivative of the general definition of **h**

$$\frac{d\mathbf{h}}{dt} = \frac{d}{dt}\left(\mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega}\right)$$

We can go ahead and do the derivative with the product rule:

$$\frac{d\mathbf{h}}{dt} = \frac{d\mathbf{r}_G}{dt} \times M\mathbf{v}_G + \mathbf{r}_G \times M\frac{d\mathbf{v}_G}{dt} + \frac{d\mathbf{I}_G}{dt}\boldsymbol{\omega} + \mathbf{I}_G\frac{d\boldsymbol{\omega}}{dt}$$

We can simplify this by noting that $d\mathbf{r}_G / dt = \mathbf{v}_G$ and of course the cross product of $\mathbf{v}_G$ with itself is zero. We can also use the definition of angular acceleration: $d\boldsymbol{\omega} / dt = \boldsymbol{\alpha}$. This gives

$$\frac{d\mathbf{h}}{dt} = \mathbf{r}_G \times M\mathbf{a}_G + \frac{d\mathbf{I}_G}{dt}\boldsymbol{\omega} + \mathbf{I}_G\boldsymbol{\alpha}$$

Finally, substitute for $d\mathbf{I}_G / dt$ from the formula in the previous section, and recall that $\mathbf{W}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}$ for all vectors **u,** and that a vector crossed with itself is zero to see that:

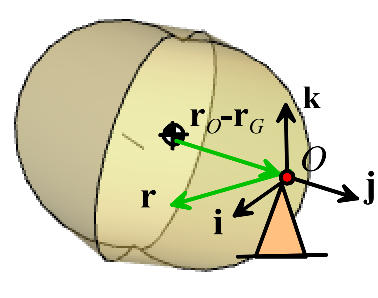
$$\begin{aligned}\frac{d\mathbf{h}}{dt} &= \mathbf{r}_G \times M\mathbf{a}_G + \left(\mathbf{W}\mathbf{I}_G - \mathbf{I}_G\mathbf{W}\right)\boldsymbol{\omega} + \mathbf{I}_G\boldsymbol{\alpha} \\ &= \mathbf{r}_G \times M\mathbf{a}_G + \mathbf{I}_G\boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{I}_G\boldsymbol{\omega} - \mathbf{I}_G\boldsymbol{\omega} \times \boldsymbol{\omega} \\ &= \mathbf{r}_G \times M\mathbf{a}_G + \mathbf{I}_G\boldsymbol{\alpha} + \boldsymbol{\omega} \times \mathbf{I}_G\boldsymbol{\omega}\end{aligned}$$

6.4.10 Special equations for angular momentum and KE of bodies that rotate about a stationary point

We often want to predict the motion of a system that rotates about a fixed pivot – a pendulum is a simple example. These problems can be be solved using a useful short-cut for the angular momentum or KE of a body rotating about a fixed point. The short-cut will give the same answer as the general formulas.

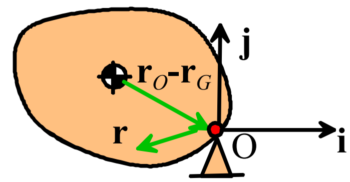
For an object that rotates about a fixed pivot at the origin:

- The total angular momentum (about the origin) is $\mathbf{h} = \mathbf{I}_O\boldsymbol{\omega}$
- The total kinetic energy is $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_O\boldsymbol{\omega}$



Here $\mathbf{I}_O$ is the mass moment of inertia about O (calculated, eg, using the parallel axis theorem). Note that the special formulas do *not* include the term involving the velocity of the COM – that's been automatically included by using $\mathbf{I}_O$ instead of $\mathbf{I}_G$.

For 2D rotation about a fixed point at the origin we can simplify these to



- The total angular momentum (about the origin) is $\mathbf{h} = I_{Ozz}\omega_z\mathbf{k}$
- The total kinetic energy is $T = \frac{1}{2}I_{Ozz}\omega_z^2$

Proof: It is straightforward to show these formulas. Let's show the two dimensional version of the kinetic energy formulas as an example. For fixed axis rotation, we can use the rigid body formulas to calculate the velocity of the center of mass (O is stationary and at the origin)

$$\mathbf{v}_G = \boldsymbol{\omega} \times \mathbf{r}_G = \omega_z \mathbf{k} \times \mathbf{r}_G$$

The general formula for kinetic energy can therefore be re-written as

$$T = \frac{1}{2}M\mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2}I_{Gzz}\omega_z^2 = \frac{1}{2}M\omega_z^2(\mathbf{k}\times\mathbf{r}_G)\cdot(\mathbf{k}\times\mathbf{r}_G) + \frac{1}{2}I_{Gzz}\omega_z^2$$

$$= \frac{1}{2}\left(M\left|\mathbf{r}_G\right|^2 + I_{Gzz}\right)\omega_z^2 = \frac{1}{2}I_{Ozz}\omega_z^2$$

The other formulas can be proved with the same method – we simply express the velocity or acceleration of the COM in the general formulas in terms of angular velocity and acceleration, and notice that we can re-arrange the result in terms of the mass moment of inertia about O.

The 3D proof is the same. Start with the general formula

$$T = \frac{1}{2}M\mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega}$$

and use the kinematics formula to find $\mathbf{v}_G$ (noting that O is stationary and at the origin)

$$\mathbf{v}_G = \boldsymbol{\omega}\times\mathbf{r}_G$$

$$T = \frac{1}{2}M\left(\boldsymbol{\omega}\times\mathbf{r}_G\right)\cdot\left(\boldsymbol{\omega}\times\mathbf{r}_G\right) + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega}$$

Remember the vector formula $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$, which shows that

$$(\boldsymbol{\omega}\times\mathbf{r}_G)\cdot(\boldsymbol{\omega}\times\mathbf{r}_G) = \left(\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right)\left(\mathbf{r}_G\cdot\mathbf{r}_G\right) - \left(\boldsymbol{\omega}\cdot\mathbf{r}_G\right)^2$$

We can re-write the kinetic energy as

$$T = \frac{1}{2}M\left((\mathbf{r}_G\cdot\mathbf{r}_G)\boldsymbol{\omega}\cdot\boldsymbol{\omega} - (\mathbf{r}_G\cdot\boldsymbol{\omega})^2\right) + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega}$$

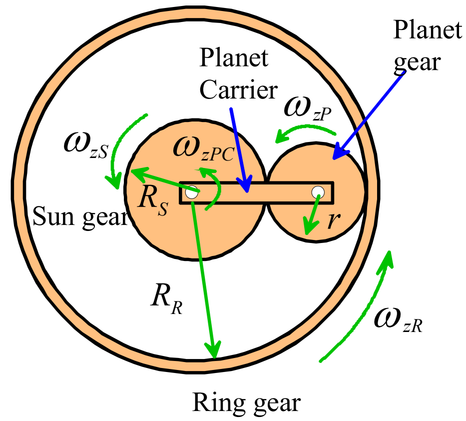
$$= \frac{1}{2}\boldsymbol{\omega}\cdot M\left[\left|\mathbf{r}_G\right|^2\mathbf{1} - \mathbf{r}_G\otimes\mathbf{r}_G\right]\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_O\boldsymbol{\omega}$$

using the parallel axis theorem.

Another way to prove the result is just to calculate the KE of the body from scratch, by summing the KE of the infinitesimal particles in the rigid body, and noting that they are all in circular motion about O.

The proof of the angular momentum formula is just the same – start with the general formula for **h** and then simplify it using $\mathbf{v}_G = \boldsymbol{\omega} \times \mathbf{r}_G$. You might like to try this as an exercise.

Example: In the planetary gear system shown, the sun gear has radius R_S and mass m, the ring gear has radius $3R_S$, while the planet gear has mass m and the planet carrier has mass $m/2$. The sun gear rotates with angular speed ω_{zS} and the ring gear is stationary.



Find a formula for the total angular momentum of the assembly about the center of the sun gear, in terms of ω_{zS}, R_S and m. Treat the gears as disks, the planet carrier as a 1D rod and assume there's only one planet gear as shown to keep things simple; this would be a rather unusual gear system but adding more gears just makes the problem tedious without illustrating any new concepts...

The 2D formula for angular momentum of a rigid body (about the origin) is

$$\mathbf{h} = \mathbf{r}_G \times m\mathbf{v}_G + I_{Gzz}\omega_z \mathbf{k}$$

where $\mathbf{r}_G$ is the position vector of the COM of the body relative to the origin.

We need to find the angular speed of all the moving parts: using the gear formulas

$$\frac{\omega_{zP} - \omega_{zPC}}{\omega_{zS} - \omega_{zPC}} = -\frac{R_S}{R_P} \qquad \frac{\omega_{zR} - \omega_{zPC}}{\omega_{zS} - \omega_{zPC}} = -\frac{R_S}{R_R} \qquad R_R = R_S + 2R_P$$

we see that

$$\frac{0 - \omega_{zPC}}{\omega_{zS} - \omega_{zPC}} = -\frac{R_S}{R_R} \Rightarrow \omega_{zS}\frac{R_S}{R_R} = \omega_{zPC}\left(1 + \frac{R_S}{R_R}\right)$$

$$\Rightarrow \omega_{zS}\frac{1}{3} = \omega_{zPC}\left(1 + \frac{1}{3}\right) \Rightarrow \omega_{zPC} = \frac{1}{4}\omega_{zS}$$

and

$$\frac{\omega_{zP} - \omega_{zPC}}{\omega_{zS} - \omega_{zPC}} = -\frac{2R_S}{R_R - R_S} \Rightarrow \frac{\omega_{zP} - \omega_{zPC}}{\omega_{zS} - \omega_{zPC}} = -1$$

$$\Rightarrow \omega_{zP} = \omega_{zPC} - (\omega_{zS} - \omega_{zPC}) = -\frac{1}{2}\omega_{zS}$$

The COM of the planet carrier is half way along its length; its COM is in circular motion with speed $V = \omega_{zPC}R_S$

Similarly the COM of the planet gear is in circular motion with speed $V = \omega_{zPC}2R_S$

Now we can add up all the angular momenta:

1. Sun $\mathbf{h}_S = \frac{1}{2} m R^2 \omega_{zS} \mathbf{k}$

2. Planet carrier $\mathbf{h}_{PC} = R_S \mathbf{i} \times (\frac{1}{2} m R_S \frac{1}{4} \omega_{Zs}) \mathbf{j} + \frac{1}{12} \frac{1}{2} m (2R_S)^2 \frac{1}{4} \omega_{zS} \mathbf{k} = \frac{1}{6} m R_S^2 \omega_{zS} \mathbf{k}$

Notice that the planet carrier rotates about the center of the sun. So, if we want, we could also use the *special formula for angular momentum of an object rotating about a fixed point*

$$\mathbf{h}_{PC} = I_{Ozz} \omega_{zPC} \mathbf{k}$$

where I_{Ozz} is the mass moment of inertia of the planet carrier about the fixed point, which must be calculated using the parallel axis theorem

$$I_{Ozz} = I_{Gzz} + Md^2 = \frac{1}{12} \frac{m}{2} (2R_s)^2 + \frac{m}{2} R_S^2 = \frac{2}{3} m R_S^2$$

(where we noted that the length of the bar is $2R_S$). We know $\omega_{zPC} = \omega_{zS} / 4$ so

$$\mathbf{h}_{PC} = I_{Ozz} \omega_{zPC} \mathbf{k} = \frac{1}{6} m R_S^2 \omega_{zS} \mathbf{k}$$

as before.

3. Planet gear $\mathbf{h}_P = 2R_S \mathbf{i} \times (m 2R_S \frac{1}{4} \omega_{Zs}) \mathbf{j} + \frac{1}{2} m (R_S)^2 (-\frac{1}{2} \omega_{zS}) \mathbf{k} = \frac{3}{4} m R_s^2 \omega_{zs} \mathbf{k}$

Note that we *can't* use the special formula for rotation about a fixed point for the planet gear, because although there is a fixed point on the planet gear (where it touches the ring), we were asked to find the angular momentum about the center of the sun. This is *not* a fixed point on the planet gear.

Sum everything $\mathbf{h} = \frac{17}{12} m R_S^2 \omega_{zS} \mathbf{k}$

6.5 Rotational forces – review of moments exerted by forces and torques

You can find a detailed discussion of forces and moments, with lots of examples, in Section 2 of these notes. Moments and torques don't come up very often in particle dynamics, but play a very important role in rigid body dynamics. We therefore review the most important concepts related to torques and moments here.

You need to remember, and understand, these ideas:

(1) A ***moment*** is a generalized force that causes an object to rotate (see section 2).

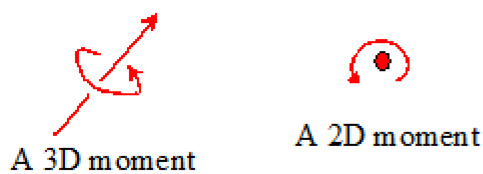
(2) ***A force can exert a moment on a rigid body***. The moment of a force (about the origin) is defined as

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

(3) In general, a force causes a rigid body to accelerate, and will also induce an angular acceleration (so it influences both translational and rotational motion).

(4) ***A 'torque' or 'pure moment' is a special kind of generalized force that causes an object to rotate, but has no effect on its translational motion***. As an example, a motor shaft (eg the bit on a power-driven screwdriver!) will exert a torque on the object connected to it.

(5) ***A torque or pure moment is a vector quantity*** – it has magnitude and direction. The direction indicates the axis associated with its rotational force (following the right hand screw convention); the magnitude represents the intensity of the rotational force. The magnitude of a torque has units of Newton Meters. A moment is often denoted by the symbols shown in the figure



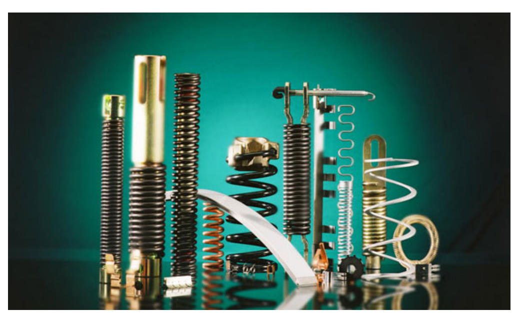
6.5.1 Rate of work done by a torque or moment: If a torque $\mathbf{Q} = Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}$ acts on an object that rotates with angular velocity $\boldsymbol{\omega}$, the rate of work done on the object by $\mathbf{Q}$ is

$$P = \mathbf{Q} \cdot \boldsymbol{\omega} = Q_x\omega_x + Q_y\omega_y + Q_z\omega_z$$

6.5.2 Torsional springs

A solid rod is a good example of a torsional spring. You could take hold of the ends of the rod and twist them, causing one end to rotate relative to the other. To do this, you would apply a *moment* or a *couple* to each end of the rod, with direction parallel to the axis of the rod. The angle of twist increases with the moment. Various torsion spring designs used in practice are shown in the picture – the image is from

http://www.mollificio.lombardo.molle.com/springs/torsion_springs.html

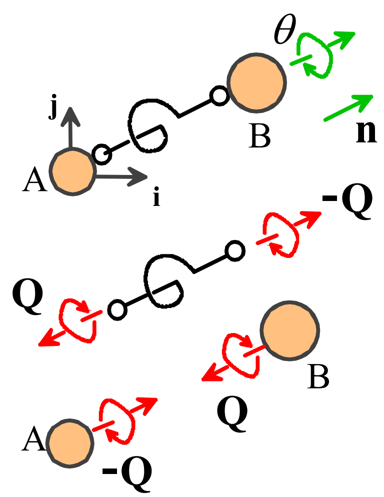


More generally, a torsional spring resists rotation, by exerting equal and opposite moments on objects connected to its ends. For a linear spring the moment is proportional to the angle of rotation applied to the spring.

The figure shows a formal free body diagram for two objects connected by a torsional spring. If object A is held fixed, and object B is rotated through an angle θ about an axis parallel to a unit vector **n**, then the spring exerts a moment

$$\mathbf{Q} = -\kappa\theta\mathbf{n}$$

on object B where κ is the *torsional stiffness* of the spring. Torsional stiffness has units of Nm/radian.



The potential energy of the moments exerted by the spring can be determined by computing the work done to twist the spring through an angle θ.

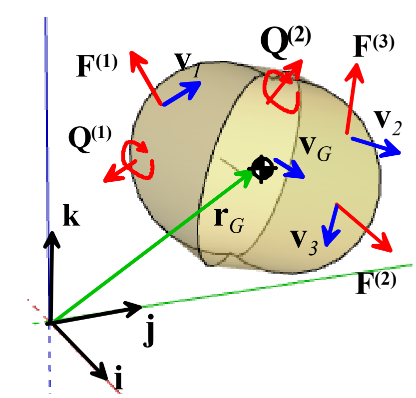
1. The work done by a moment **Q** due to twisting through a very small angle $d\theta$ about an axis parallel to a vector **n** is
$$dW = \mathbf{Q} \cdot d\theta\mathbf{n}$$
2. The potential energy is the negative of the total work done by **M,** i.e.

$$V = -\int_0^\theta \mathbf{Q} \cdot d\theta\mathbf{n} = -\int_0^\theta (-\kappa\theta\mathbf{n}) \cdot d\theta\mathbf{n} = \int_0^\theta \kappa\theta d\theta = \frac{1}{2}\kappa\theta^2$$

A potential energy cannot usually be defined for most concentrated moments, because rotational motion is itself path dependent (the orientation of an object that is given two successive rotations depends on the order in which the rotations are applied).

6.6 Dynamics of rigid bodies

We predict the position and velocity of a particle by integrating **F**=m**a**. For a rigid body, we need to predict both its position and orientation. We use the following equations to do this.



The figure shows a rigid body subjected to several forces $\mathbf{F}^{(i)}$ and torques (pure moments) $\mathbf{Q}^{(i)}$. During a representative time interval $t_0 < t < t_1$ the forces exert a linear impulse $\Im$ and angular impulse **A**, and do total work on the rigid body ΔW.

The body has total mass M and mass moment of inertia $\mathbf{I}_G$ about the center of mass.

Let $\mathbf{r}_G, \mathbf{v}_G, \mathbf{a}_G$ denote the position, velocity and acceleration of the center of mass, and let $\boldsymbol{\omega}, \boldsymbol{\alpha}$ denote the angular velocity and acceleration.

The linear and angular momentum (about the origin) of the rigid body follow as $\mathbf{p} = M\mathbf{v}_G$, $\mathbf{h} = M\mathbf{r}_G \times \mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega}$, and its kinetic energy is $T = \frac{1}{2}M\left|\mathbf{v}_G\right|^2 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}_G\boldsymbol{\omega}$.

The equations of motion are then

Force-acceleration relation: $\sum_i \mathbf{F}^{(i)} = M\mathbf{a}_G$

Moment – angular velocity/acceleration relation $\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = M\mathbf{r}_G \times \mathbf{a}_G + \mathbf{I}_G\boldsymbol{\alpha} + \boldsymbol{\omega}\times\left[\mathbf{I}_G\boldsymbol{\omega}\right]$

Force-momentum and impulse-momentum relation: $\sum_i \mathbf{F}^{(i)} = \frac{d\mathbf{p}}{dt}$ $\qquad \Im = \mathbf{p}_1 - \mathbf{p}_0$

Moment – angular momentum relation: $\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = \frac{d\mathbf{h}}{dt}$ $\qquad \mathbf{A} = \mathbf{h}_1 - \mathbf{h}_0$

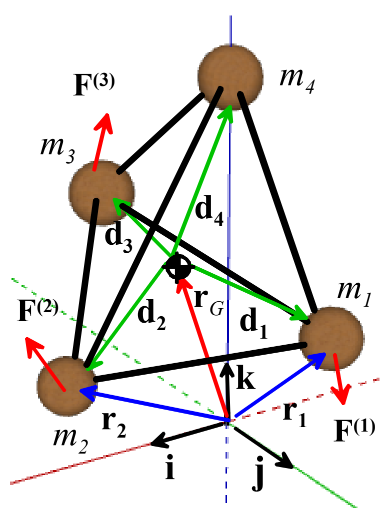
Power – work – kinetic energy relation $\sum_i \mathbf{F}^{(i)}\cdot\mathbf{v}^{(i)} + \sum_j \mathbf{Q}^{(j)}\cdot\boldsymbol{\omega} = \frac{dT}{dt}$ $\qquad \Delta W = T_1 - T_0$

For **2D planar motion** we can use the simplified formulas

$$\sum_i \mathbf{F}^{(i)} = M\mathbf{a}_G$$

$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = M\mathbf{r}_G \times \mathbf{a}_G + I_{Gzz}\alpha_z\mathbf{k}$$

Derivations: It is possible to obtain the equations of motion for a rigid body from Newton's laws for a particle – the basic idea is to assume that a rigid body consists of an infinite number of particles connected by rigid massless links – but this isn't really a rigorous proof, because we have to assume that the links are two-force members, and there is no way to prove that this is a realistic description of matter. Another viewpoint is to accept conservation of linear momentum and angular momentum as two separate physical laws (the linear momentum is just Newton's law, and the angular momentum equation is sometimes referred to as Euler's law). We can then 'prove' that a rigid body can be represented as a bunch of particles connected by two force members. We'll show the first approach here.



The figure shows a system of particles connected by rigid massless links.

The length of the link between the ith and jth particle will be denoted by L_{ij} We assume that all the links are two-force members.

The particles are subjected to a set of external forces $\mathbf{F}^{(i)}$. We denote the magnitude of the force in the member connecting the *i*th and *j*th particle by R_{ij} (by convention a positive R_{ij} represents an attractive force between the particles). Note that the $R_{ij} = R_{ji}$ because the two particles exert equal and opposite forces on each other. The vector valued force exerted on the *ith* particle by the *j*th follows as

$$\mathbf{R}_{ij} = R_{ij} \frac{\mathbf{r}_j - \mathbf{r}_i}{L_{ij}}$$

(to see this note that $(\mathbf{r}_j - \mathbf{r}_i) / L_{ij}$ is a unit vector from the ith to the jth particles)

We can start the derivation with the force-linear momentum relation for a single particle. For example, for the *i*th particle (see section 4 of the notes)

$$\mathbf{F}^i + \sum_{j \neq i} R_{ij} \frac{(\mathbf{r}_j - \mathbf{r}_i)}{L_{ij}} = \frac{d}{dt} m_i \mathbf{v}_i$$

Sum this over all particles

$$\sum_i \mathbf{F}^i + \sum_i \sum_{j \neq i} R_{ij} \frac{(\mathbf{r}_j - \mathbf{r}_i)}{L_{ij}} = \sum_i \frac{d}{dt} m_i \mathbf{r}_i$$

But we know that $\sum_i m_i \mathbf{v}_i = M \mathbf{v}_G$, and since $R_{ij} = R_{ji}$, $L_{ij} = L_{ji}$ the second term on the left hand side is zero. Therefore

$$\sum_i \mathbf{F}^i = M \frac{d\mathbf{v}_G}{dt} = M \mathbf{a}_G = \frac{d\mathbf{p}}{dt}$$

Since this is independent of the number of particles, it must also apply to a rigid body. This shows that the force-momentum and force-acceleration for a rigid body can be derived from Newton's law for a particle.

We can derive the angular momentum relation for a rigid body using the same idea. For one particle we have the angular momentum equation

$$\mathbf{r}_i \times \mathbf{F}^{(i)} + \mathbf{r}_i \times \sum_{j \neq i} R_{ij} \frac{(\mathbf{r}_j - \mathbf{r}_i)}{L_{ij}} = \mathbf{r}_i \times \mathbf{F}^{(i)} + \mathbf{r}_i \times \sum_{j \neq i} R_{ij} \frac{\mathbf{r}_j}{L_{ij}} = \frac{d\mathbf{h}_i}{dt}$$

where we have noted that $\mathbf{r}_i \times \mathbf{r}_i = \mathbf{0}$. We can sum this over all the particles

$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_i \sum_{j \neq i} R_{ij} \frac{\mathbf{r}_i \times \mathbf{r}_j}{L_{ij}} = \frac{d}{dt} \sum_i \mathbf{h}_i$$

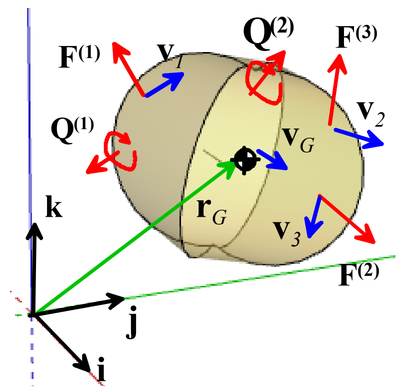
The second term here is zero, because $\mathbf{r}_i \times \mathbf{r}_j = -\mathbf{r}_j \times \mathbf{r}_i$ and $R_{ij} = R_{ji}$ $L_{ij} = L_{ji}$ (just write out the sum term by term for some finite number of particles – eg two – if you don't see this). The term on the right hand side is clearly just the total angular momentum of the system. If we replace some subset of the forces with a statically equivalent torque and force, we obtain the moment-angular momentum equation.

6.7 Summary of equations of motion for rigid bodies

In this section, we collect together all the important formulas from the preceding sections, and summarize the equations that we use to analyze motion of a rigid body.

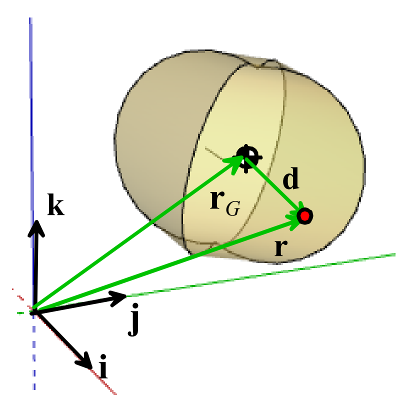
We consider motion of a rigid body that has mass density ρ during some time interval $t_0 < t < t_1$, and define the following quantities:

6.7.1 Forces, torques, impulse, work, power



- The total force acting on the body $\sum_i \mathbf{F}^{(i)}$
- The total linear impulse exerted by forces during the time interval $\Im = \int_{t_0}^{t_1} \sum_i \mathbf{F}^{(i)}(t)dt$
- The total moment (including torques) acting on the body $\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)}$
- The tot al angular impulse exerted on the body during the time interval $\mathbf{A} = \int_{t_0}^{t_1} \left(\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)}(t) + \sum_j \mathbf{Q}^{(j)} \right) dt$
- The rate of work done by forces and torques acting on the body $P = \sum_i \mathbf{F}^{(i)} \cdot \mathbf{v}_i + \sum_j \mathbf{Q}^{(j)} \cdot \boldsymbol{\omega}$
- The total work done by forces and torques on the body during the time interval $W = \int_{t_0}^{t_1} P(t)dt$

6.7.2 Inertial properties

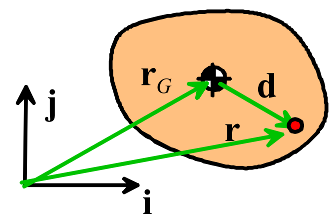


- The total mass is $M = \int_V \rho dV$
- The position of the center of mass is $\mathbf{r}_G = \frac{1}{M} \int_V \mathbf{r} \rho dV$
- The mass moment of inertia about the center of mass

$$\mathbf{I}_G = \int_V \rho \begin{bmatrix} d_y^2 + d_z^2 & -d_x d_y & -d_x d_z \\ -d_x d_y & d_x^2 + d_z^2 & -d_y d_z \\ -d_x d_z & -d_y d_z & d_x^2 + d_y^2 \end{bmatrix} dV$$

where $\mathbf{d} = \mathbf{r} - \mathbf{r}_G$

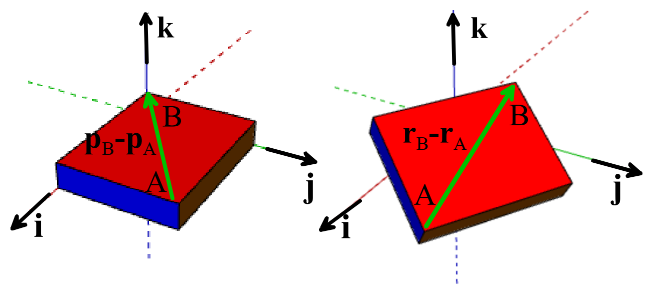
For a 2D body with mass per unit area μ we use



- The total mass is $M = \int_A \mu dA$
- The position of the center of mass is $\mathbf{r}_G = \frac{1}{M}\int_A \mathbf{r}\mu dA$
- The mass moment of inertia about the center of mass is $I_{Gzz} = \frac{1}{M}\int_A \mu(d_x^2 + d_y^2)dA$

where $\mathbf{d} = \mathbf{r} - \mathbf{r}_G$

6.7.3 Describing motion



- The rotation tensor (matrix) maps the vector connecting two points in a solid before it moves to its position after motion

$$\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)$$

- The spin tensor is related to $\mathbf{R}$ by

$$\mathbf{W} = \frac{d\mathbf{R}}{dt}\mathbf{R}^T \qquad \frac{d\mathbf{R}}{dt} = \mathbf{W}\mathbf{R}$$

- Rotation through an angle θ about an axis parallel to a unit vector $\mathbf{n} = n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k}$ is

$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & (1-\cos\theta)n_y n_z - \sin\theta n_x \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & (1-\cos\theta)n_y n_z + \sin\theta n_x & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$

- The angular velocity vector $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$ is related to $\mathbf{W}$ by

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

- The angular acceleration vector is $\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$

- The velocities of two points A and B in a rotating rigid body are related by $\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)$

- The accelerations of A and B are related by

$$\mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times (\mathbf{v}_B - \mathbf{v}_A) = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times \left[\boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)\right]$$

6.7.4 Momentum and Energy

- The total linear momentum is $\mathbf{p} = M\mathbf{v}_G$
- The angular momentum (about the origin) is $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + \mathbf{I}_G\boldsymbol{\omega}$
- The total kinetic energy is $T = \frac{1}{2}M\mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_G\boldsymbol{\omega}$

For 2D planar problems, we know $\boldsymbol{\omega} = \omega_z \mathbf{k}$. In this case, we can use

- The total linear momentum is $\mathbf{p} = M\mathbf{v}_G$
- The total angular momentum (about the origin) is $\mathbf{h} = \mathbf{r}_G \times M\mathbf{v}_G + I_{Gzz}\omega_z \mathbf{k}$
- The total kinetic energy is $T = \frac{1}{2}M\left|\mathbf{v}_G\right|^2 + \frac{1}{2}I_{Gzz}\omega_z^2$

6.7.5 Conservation laws

- Linear momentum $\sum_i \mathbf{F}^{(i)} = \frac{d\mathbf{p}}{dt}$ $\quad \Im = \mathbf{p}_1 - \mathbf{p}_0$
- Angular momentum $\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = \frac{d\mathbf{h}}{dt}$ $\quad \mathbf{A} = \mathbf{h}_1 - \mathbf{h}_0$
- Work-Power - Kinetic Energy relation $\sum_i \mathbf{F}^{(i)} \cdot \mathbf{v}^{(i)} + \sum_j \mathbf{Q}^{(j)} \cdot \boldsymbol{\omega} = \frac{dT}{dt}$ $\quad \Delta W = T_1 - T_0$
- Energy equation for a conservative system $\frac{d}{dt}(T + V) = 0$ $\quad T_0 + V_0 = T_1 + V_1$

6.7.6 Linear and angular momentum equations in terms of accelerations

The linear and angular momentum conservation equations can also be expressed in terms of accelerations, angular accelerations, and angular velocities. The results are

$$\sum_i \mathbf{F}^{(i)} = M\mathbf{a}_G$$

$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = M\mathbf{r}_G \times \mathbf{a}_G + \mathbf{I}_G\boldsymbol{\alpha} + \boldsymbol{\omega} \times [\mathbf{I}_G\boldsymbol{\omega}]$$

For **2D planar motion** we can use the simplified formulas

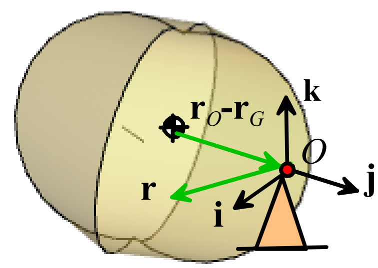
$$\sum_i \mathbf{F}^{(i)} = M\mathbf{a}_G$$

$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = M\mathbf{r}_G \times \mathbf{a}_G + I_{Gzz}\alpha_z\mathbf{k}$$

6.7.7 Special equations for analyzing bodies that rotate about a stationary point

We often want to predict the motion of a system that rotates about a fixed pivot – a pendulum is a simple example. These problems can be solved using the equations in 6.6.5 and 6.6.6, but can also be solved using a useful short-cut.

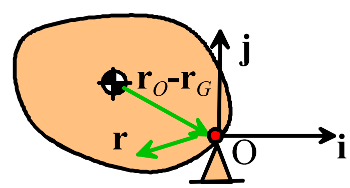
For an object that rotates about a fixed pivot at the origin:



- The total angular momentum (about the origin) is $\mathbf{h} = \mathbf{I}_O\boldsymbol{\omega}$
- The total kinetic energy is $T = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}_O\boldsymbol{\omega}$
- The equation of rotational motion is
$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = \mathbf{I}_O\boldsymbol{\alpha} + \boldsymbol{\omega} \times [\mathbf{I}_O\boldsymbol{\omega}]$$

Here $\mathbf{I}_O$ is the mass moment of inertia about O (calculated, eg, using the parallel axis theorem)

For 2D rotation about a fixed point at the origin we can simplify these to



- The total angular momentum (about the origin) is $\mathbf{h} = I_{Ozz}\omega_z\mathbf{k}$
- The total kinetic energy is $T = \frac{1}{2}I_{Ozz}\omega_z^2$
- The equation of rotational motion is
$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = I_{Ozz}\alpha_z\mathbf{k}$$

Proof: It is straightforward to show these formulas. Let's show the two dimensional version of the kinetic energy formulas as an example. For fixed axis rotation, we can use the rigid body formulas to calculate the velocity of the center of mass (O is stationary and at the origin)

$$\mathbf{v}_G = \boldsymbol{\omega} \times \mathbf{r}_G = \omega_z \mathbf{k} \times \mathbf{r}_G$$

The general formula for kinetic energy can therefore be re-written as

$$T = \frac{1}{2} M \mathbf{v}_G \cdot \mathbf{v}_G + \frac{1}{2} I_{Gzz} \omega_z^2 = \frac{1}{2} M \omega_z^2 (\mathbf{k} \times \mathbf{r}_G) \cdot (\mathbf{k} \times \mathbf{r}_G) + \frac{1}{2} I_{Gzz} \omega_z^2$$

$$= \frac{1}{2} \left(M \left| \mathbf{r}_G \right|^2 + I_{Gzz} \right) \omega_z^2 = \frac{1}{2} I_{Ozz} \omega_z^2$$

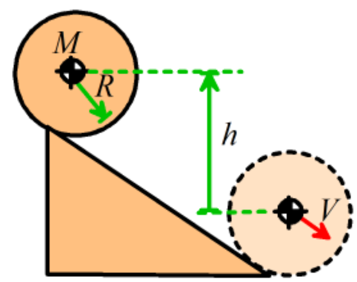
The other formulas can be proved with the same method – we simply express the velocity or acceleration of the COM in the general formulas in terms of angular velocity and acceleration, and notice that we can re-arrange the result in terms of the mass moment of inertia about O.

6.8 Examples of solutions to problems involving motion of rigid bodies

The best way to learn how to use the equations in section 6.6 is just to work through a series of examples.

6.8.1 Solutions to 2D problems

Example 1: A solid of revolution (eg a cylinder or sphere) with mass M and mass moment of inertia about its COM I_{Gzz} is released from rest at the top of a ramp. It rolls without slip. Calculate its velocity at the bottom of the ramp.



- The system is conservative, so we can solve the problem using energy conservation. The energy equation tells us that the sum of kinetic and potential energy of the cylinder is constant: $T_0 + V_0 = T_1 + V_1$
- We can take the datum for potential energy to be the position of the COM at the bottom of the ramp. The initial potential energy is therefore $V_0 = Mgh$; the final potential energy is zero.
- The initial kinetic energy is zero, because the cylinder is stationary. The final kinetic energy is $T = \frac{1}{2} M v_x^2 + \frac{1}{2} I_{Gzz} \omega_z^2$.
- The energy equation gives $T_0 + V_0 = T_1 + V_1 \Rightarrow Mgh = \frac{1}{2} M v_x^2 + \frac{1}{2} I_{Gzz} \omega_z^2$
- Finally, since the cylinder rolls without slip, we know that $v_x = -R\omega_z$.

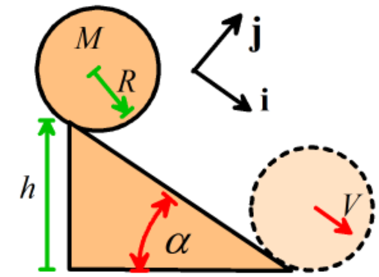
Hence

$$2Mgh = Mv_x^2 + \frac{I_{Gzz}}{R^2} v_x^2$$

$$\Rightarrow v_x = \sqrt{\frac{2gh}{1 + I_{Gzz} / (MR^2)}}$$

This formula predicts that an object with a smaller inertia I_{Gzz} will move faster than an object with a large inertia. A sphere rolls down the ramp more quickly than a cylinder, for example, and a solid cylinder rolls more quickly than a ring.

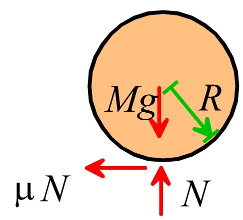
Example 2: For the problem treated in the preceding section, calculate the critical value of friction coefficient necessary to prevent slip at the contact.



If we want to learn about forces, we have to use the linear and angular momentum equations. This problem can be solved with the 2D formulas in terms of accelerations:

$$\sum_i \mathbf{F}^{(i)} = M\mathbf{a}_G$$

$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = M\mathbf{r}_G \times \mathbf{a}_G + I_{Gzz}\alpha_z \mathbf{k}$$



- The figure shows a free body diagram for the cylinder (or sphere)
- We know that the COM is always a constant height above the ramp, so the acceleration must be parallel to **i**. The linear momentum equation gives

$$(Mg\sin\alpha - T)\mathbf{i} + (N - Mg\cos\alpha)\mathbf{j} = Ma_{Gx}\mathbf{i}$$

- We can use the angular momentum equation – it is convenient to take moments about the contact point C. (There are no torques in this problem).

$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = M\mathbf{r}_G \times \mathbf{a}_G + I_{Gzz}\alpha_z \mathbf{k}$$

$$\Rightarrow -RMg\sin\alpha\mathbf{k} = R\mathbf{j} \times a_{Gx}\mathbf{i} + I_{Gzz}\alpha_z\mathbf{k} = -MRa_{Gx}\mathbf{k} + I_{Gzz}\alpha_z\mathbf{k}$$

- Finally, we can use the rolling wheel formula for accelerations $a_{Gx} = -R\alpha_z$.
- The preceding results give:

$$-RMg\sin\alpha = -MRa_{Gx} - I_{Gzz}\frac{a_{Gx}}{R}$$

$$\Rightarrow a_{Gx} = \frac{MgR\sin\alpha}{MR + (I_{Gzz}/R)} = \frac{g\sin\alpha}{1 + I_{Gzz}/(MR^2)}$$

- Finally, substituting back into the **i** components of (1):

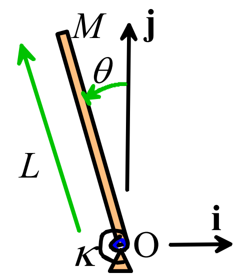
$$T = Mg\sin\alpha - Ma_{Gx}$$

$$= Mg\sin\alpha - \frac{Mg\sin\alpha}{1 + I_{Gzz}/(MR^2)} = \frac{I_{Gzz}/(MR^2)}{1 + I_{Gzz}/(MR^2)} Mg\sin\alpha$$

- The **j** component of (1) gives $N = Mg\cos\alpha$
- For no slip $|T| \le \mu N \Rightarrow \quad \mu \ge \dfrac{I_{Gzz}/(MR^2)}{1 + I_{Gzz}/(MR^2)}\tan\alpha$

The formula shows that objects with large values of I_{Gzz} / MR^2 are more likely to slip. If the inertia is very small, slip will never occur. A ring will slip on a lower slope than a cylinder, which will slip on a lower slope than a sphere.

Example 3: A vertical mast can be idealized as a slender rod with length L and mass M, which is held in an inverted position by a torsional spring with stiffness κ at its base. Find the equation of motion for the angle θ in the figure, and hence determine the natural frequency of vibration of the mast.



This is a conservative system. Also, the mast rotates about a fixed point. We can analyze the problem using energy methods, and use the special formulas for rotation about a fixed point.

- The kinetic energy formula for planar motion is

$$T = \frac{1}{2} I_{Ozz} \omega_z^2$$

- For planar motion we know that

$$\omega_z = \frac{d\theta}{dt}$$

- We can use the parallel axis theorem to calculate the mass moment of inertia of a rod about one end:

$$I_{Ozz} = I_{Gzz} + Md^2 = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2$$

- Gravity and the torsional spring both contribute to the total potential energy of the system. The total potential energy is

$$V = Mg\frac{L}{2}\cos\theta + \frac{1}{2}\kappa\theta^2$$

- Energy conservation means that

$$T + V = const \Rightarrow \frac{d}{dt}(T + V) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2} I_{Ozz}\omega_z^2 + MgL\cos\theta + \frac{1}{2}\kappa\theta^2\right) = 0$$

$$\Rightarrow I_{Ozz}\frac{d\omega_z}{dt}\omega_z - MgL\sin\theta\frac{d\theta}{dt} + \kappa\theta\frac{d\theta}{dt} = 0$$

- Recall that $\omega_z = \dfrac{d\theta}{dt}$ so

$$\Rightarrow I_{Ozz}\frac{d^2\theta}{dt^2} - MgL\sin\theta + \kappa\theta = 0$$

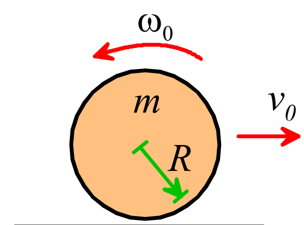
- We assume that θ is small enough that $\sin\theta \approx \theta$, so

$$\Rightarrow \frac{I_{Ozz}}{\kappa - MgL}\frac{d^2\theta}{dt^2} + \theta = 0$$

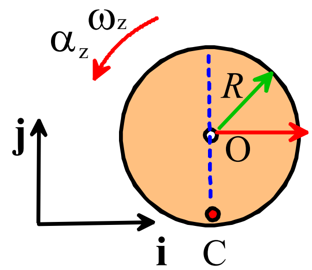
- This is a standard 'Case I' undamped vibration EOM, so we can just read off the natural frequency

$$\omega_n = \sqrt{\frac{\kappa - MgL}{I_{Ozz}}} = \sqrt{\frac{3(\kappa - MgL)}{ML^2}}$$

Example 4: A thin uniform disk of radius R, mass m and mass moment of inertia $mR^2/2$ is placed on the ground with a positive velocity v_0 in the horizontal direction, and a counterclockwise rotational velocity (a backspin) ω_0. The contact between the disk and the ground has friction coefficient μ. The disk initially slips on the ground, and for a suitable range of values of ω_0 and v_0 its direction of motion may reverse. The goal of this problem is to calculate the conditions where this reversal will occur.



General discussion of slipping contacts: Solving problems with sliding at a contact is always tricky, because we have to draw the friction forces in the correct direction. Before tackling the example, we will summarize the general rules. We will consider a wheel as an example, but the rules apply to contact between any object and a stationary surface. The figure shows a wheel that spins with angular velocity $\boldsymbol{\omega} = \omega_z \mathbf{k}$ while the center moves with speed $\mathbf{v}_O = v_{ox}\mathbf{i}$. The direction of the friction force is determined by the direction of motion of the point on the wheel that instantaneously touches the ground, which can be calculated from the formula



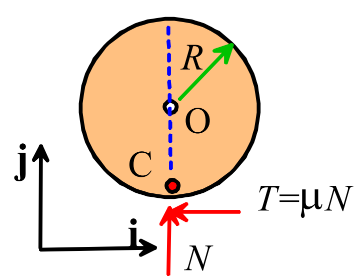
$$\mathbf{v}_C = (v_{Ox} + \omega_z R)\mathbf{i}$$

Friction always acts to try to bring point C to rest – if C is moving to the right, friction acts to the left; if C is moving to the left, friction acts to the right.

There are three possible cases:

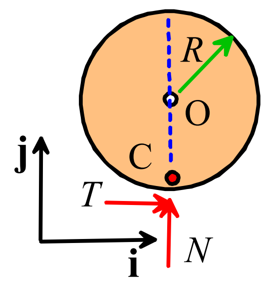
Forward slip: $v_{Ox} + \omega_z R > 0$ Point C moves in the positive **i** direction over the ground

- Slip occurs at the contact,
- We have to use the friction law $T = \mu N$
- Point C is moving to the right, so friction must act to the left



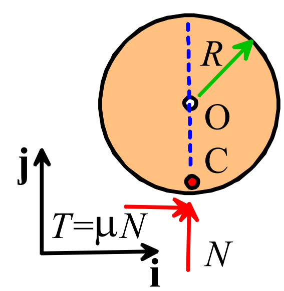
Pure rolling $v_{Ox} + \omega_z R = 0$ **.** Point C is stationary.

- No slip occurs at the contact.
- In this case $|T| < \mu N$
- We can draw the friction force in either direction at the contact (if we choose the wrong direction, our calculations will just tell us that T is negative). It is usually convenient to choose T to act in the positive **i** direction, but this is not necessary.



Reverse slip: $v_{Ox} + \omega_z R < 0$ Point C moves in the negative **i** direction over the ground

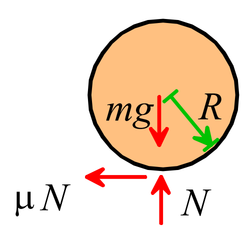
- Slip occurs at the contact,
- We have to use the friction law $T = \mu N$
- Point C is moving to the left, so friction must act to the right



Now we return to the example.

4.1 Draw a free body diagram showing the forces acting on the disk just after it hits the ground.

We are given that v_{x0} and ω_{z0} are both positive so we have $v_{Ox} + \omega_z R > 0$ **.** This is forward slip, and we use the corresponding FBD.



4.2 Hence, find formulas for the initial acceleration a and angular acceleration α for the disk, in terms of g, R and μ. Note that the contact point is slipping.

The equations of motion are

$$\sum_i \mathbf{F}^{(i)} = M\mathbf{a}_G \qquad \sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = M\mathbf{r}_G \times \mathbf{a}_G + I_{Gzz}\alpha_z \mathbf{k}$$

$$-\mu N\mathbf{i} + (N - mg)\mathbf{j} = ma_x\mathbf{i}$$
$$-\mu NR\mathbf{k} = I_{Gzz}\alpha_z\mathbf{k}$$

Solving these and using $I_{Gzz} = mR^2 / 2$:

$$N = mg \qquad \mu N = -ma_x \qquad \frac{1}{2}mR^2\alpha_z = -\mu NR$$
$$\Rightarrow a_x = -\mu g \qquad \alpha_z = -2\mu g / R$$

4.3 Find formulas for the velocity and angular velocity of the disk, during the period while the contact point is still slipping.

The acceleration and angular acceleration are constant, so we can use the constant acceleration formulas:

$$v_x = v_0 - \mu gt \qquad \omega_z = \omega_0 - 2\mu gt / R$$

4.4 Find a formula for the time at which the disk will reverse its direction of motion.

Velocity is reversed where *v=0*. From the previous part, $v = v_0 - \mu gt \Rightarrow t = v_0 / \mu g$ at the reversal.

4.5 Find a formula for the time at which the disk begins to roll on the ground without slip. Hence, show that the disk will reverse its direction only if $v_0 < \omega_0 R / 2$

Rolling without sliding starts when $v_{xO} = -\omega_z R$. We have that

$$\omega_z = \omega_0 - 2\mu gt / R \qquad v_x = v_0 - \mu gt$$
$$\Rightarrow v_0 - \mu gt = -\left(\omega_0 R - 2\mu gt\right) \qquad \text{when } v_x = -\omega_z R$$
$$\Rightarrow t = (v_0 + \omega_0 R) / 3\mu g$$

The reversal will only occur if rolling without slip occurs after the reversal of velocity. This means

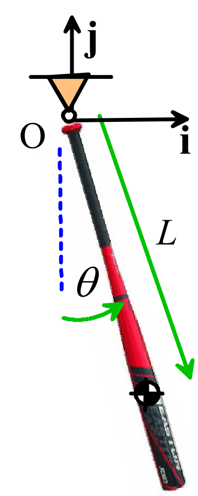
$$(v_0 + \omega_0 R)/3\mu g > v_0/\mu g \Rightarrow v_0 < \omega_0 R/2$$

Example 5: The 'Sweet Spot' on a softball or baseball bat, or tennis or squash racket is a point that minimizes the reaction forces acting on the athlete's hand when the ball is struck. In fact, any rigid body has a sweet spot – the magic point is called the 'center of percussion' of a rigid body.

For baseball and softball bats in particular, there is a standard ASTM test that can be used to measure the position of the sweet spot. The test works like this: the bat is suspended from the knob on handle, so it swings like a pendulum. The period of vibration τ of the swinging bat is then measured. ASTM say that the center of percussion is then a distance

$$d = \frac{\tau^2 g}{4\pi^2}$$

from the end of the handle. Why does this work? It seems that this test has nothing whatever to do with a ball hitting the bat!



We will solve this problem in two parts. First, we will calculate a formula for the period of vibration in the ASTM test. Then we will calculate the position of the center of percussion. We will see that the ASTM test does indeed make the correct prediction.

We can calculate the period using the energy method. The figure shows the ASTM pendulum test. We assume that

- The bat has a mass moment of inertia about its COM I_{Gzz}
- The COM is a distance L from O

The bat pivots about O, so we can use the fixed axis rotation formula for the kinetic energy

$$T = \frac{1}{2} I_{Ozz} \left(\frac{d\theta}{dt} \right)^2$$

Here $I_{Ozz} = I_{Gzz} + ML^2$ (using the parallel axis theorem).

The potential energy is $V = -MgL\cos\theta$.

Energy conservation gives

$$T + V = \text{constant} \Rightarrow \frac{d}{dt}(T + V) = 0$$

$$\Rightarrow I_{Ozz} \frac{d^2\theta}{dt^2}\frac{d\theta}{dt} + MgL\sin\theta \frac{d\theta}{dt} = 0$$

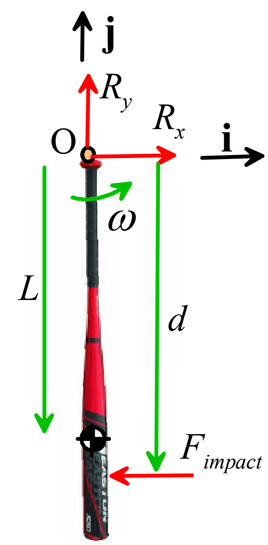
If θ is small then $\sin\theta \approx \theta$ so the equation of motion reduces to

$$\frac{I_{Ozz}}{MgL}\frac{d^2\theta}{dt^2} + \theta = 0$$

This is a standard 'Case 1' EOM. The natural frequency is $\omega_n = \sqrt{\frac{MgL}{I_{Ozz}}}$ so the period is

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I_{Ozz}}{MgL}}$$

Next, we find the position of the 'sweet spot'. We can do this by calculating the reaction forces on the handle when the bat is struck, and finding the impact point that minimizes the reaction force.



The figure shows an impact event. We assume that:

- The bat rotates in the horizontal plane (so gravity acts out of the plane of the figure).
- The bat rotates about the point O
- The ball impacts the bat a distance d from the handle.
- The ball exerts a (large) force F_{impact} on the bat
- Reaction forces R_x, R_y act on the handle during the impact.

This is a planar problem, so we can use the 2D equations of motion. The equation for translational motion gives
$(R_x - F_{impact})\mathbf{i} + R_y\mathbf{j} = M\mathbf{a}_G$

For the rotational equation we can also use the short-cut for fixed axis rotation

$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = I_{Ozz}\alpha_z\mathbf{k}$$

$$\Rightarrow -F_{impact}d\mathbf{k} = I_{Ozz}\alpha_z\mathbf{k} \qquad \Rightarrow \alpha_z = -\frac{F_{impact}d}{I_{Ozz}}$$

We can relate $\mathbf{a}_G$ to α_z using the rigid body formula:

$$\mathbf{a}_G = \alpha_z\mathbf{k}\times\mathbf{r}_G - \omega_z^2\mathbf{r}_G = \alpha_z L\mathbf{i} + \omega_z^2 L\mathbf{j}$$

We therefore see that

$$(R_x - F_{impact})\mathbf{i} + R_y\mathbf{j} = M\left(\alpha_z L\mathbf{i} + \omega_z^2 L\mathbf{j}\right)$$

$$\Rightarrow R_x = F_{impact} + M\alpha_z L$$

$$\Rightarrow R_x = F_{impact}\left(1 - \frac{MdL}{I_{Ozz}}\right)$$

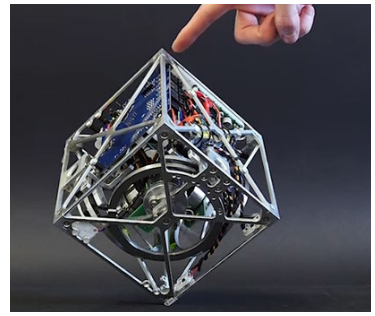
The sweet spot is at the position that makes $R_x = 0$, which shows that

$$d = \frac{I_{Ozz}}{ML}$$

For comparison, the ASTM formula gives

$$d = \frac{\tau^2 g}{4\pi^2} = \frac{g}{4\pi^2}\left(2\pi\sqrt{\frac{I_{Ozz}}{MgL}}\right)^2 = \frac{I_{Ozz}}{ML}$$

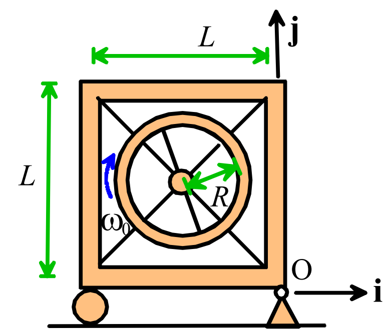
Example 6. The 'Cubli' is used to develop control algorithms used to stabilize aircraft and spacecraft. It consists of a cube whose attitude can be controlled by spinning a set of reaction wheels inside the cube.



This simplified 1-D version of the device is used to test the algorithm that stands the cube up on one edge. The goal of this problem is to do the preliminary design calculations needed to set up the system.

Idealize the rectangular frame as four rods with length L and combined mass M and the spinning wheel as a ring with radius R and mass m. The corner at O is supported by a frictionless bearing.

Part 1: Find formulas for the mass moments of inertia of the frame and the wheel (about the center of the wheel).



The ring is easy – we can use the formula $I_R = mR^2$

The frame is made up of four rods of mass *M/4*. The moment of inertia of one rod about its center of mass is $\frac{1}{12}\frac{M}{4}L^2$. We need to shift the COM by a distance of *L/2* to the center of the frame. The total mass moment of inertia of the frame is therefore

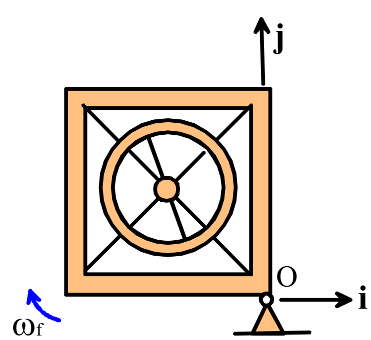
$$I_F = 4\left(\frac{1}{12}\frac{M}{4}L^2 + \frac{M}{4}\left(\frac{L}{2}\right)^2\right) = \frac{1}{3}ML^2$$

Part 2: The frame is at rest and the wheel is spun up (clockwise) to an angular speed ω_0 . Find the total angular momentum of the system about the corner at O.

The formula for angular momentum is $\mathbf{h}_O = \sum \mathbf{r} \times m\mathbf{v}_G + \sum \mathbf{I}\boldsymbol{\omega}$

Since the frame is not moving only the second term contributes and we get $\mathbf{h} = -mR^2\omega_0\mathbf{k}$

Part 3: Thee wheel is then braked quickly, which causes the frame to rotate about the corner O at angular speed ω_f , while the motor driving the ring spins at (clockwise) angular speed ω_1 (note that this is relative to the frame). Write down the angular momentum of the system about



O.

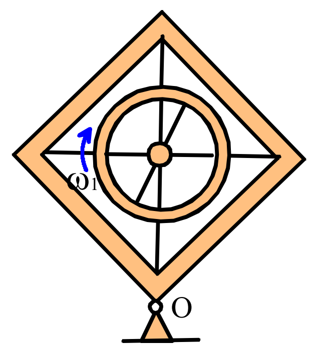
Note that the frame rotates about O so the COM of the ring and frame are both in circular motion about O. We know the speed of their COMs are therefore $\omega_f L/\sqrt{2}$

Use the formula again

$$
\begin{aligned}
\mathbf{h} &= \sum \mathbf{r}_G \times m\mathbf{v}_G + I_{Gzz}\omega_z \mathbf{k} \\
&= -\left(\frac{1}{3} ML^2 \omega_f + \frac{L}{\sqrt{2}} M \frac{L}{\sqrt{2}} \omega_f + mR^2(\omega_1 + \omega_f) + \frac{L}{\sqrt{2}} m \frac{L}{\sqrt{2}} \omega_f \right) \mathbf{k} \\
&= -\left(\frac{5}{6} ML^2 + m(R^2 + \frac{L^2}{2}) \right) \omega_f \mathbf{k} - mR^2 \omega_1 \mathbf{k}
\end{aligned}
$$

We could also use the fixed axis rotation formula for the frame (using the mass moment of inertia about O) but this would not work for the ring, because O is not a stationary point on the ring.

Part 4: Explain why angular momentum is conserved about O during the braking. Use momentum conservation to find an equation relating ω_f to $(\omega_1 - \omega_0)$



The external forces acting on the frame and ring together are (1) gravity and (2) reaction forces at O. We assume that the speed change of the rotor takes place over a very short time interval. The force of gravity is constant and exerts a negligible impulse on the system during this time interval. The reactions exert a finite impulse, but if we take moments about O the external angular impulse about O on the system vanishes. This means angular momentum must be conserved.

$$
\mathbf{h}_1 - \mathbf{h}_0 = 0 \Rightarrow -\left(\frac{5}{6} ML^2 + m(R^2 + \frac{L^2}{2}) \right) \omega_f \mathbf{k} - mR^2 \omega_1 \mathbf{k} + mR^2 \omega_0 \mathbf{k} = 0
$$

$$
\Rightarrow \omega_f = \frac{mR^2(\omega_0 - \omega_1)}{\left(\frac{5}{6} ML^2 + m(R^2 + \frac{L^2}{2}) \right)}
$$

Part 5: For the special case $\omega_1 = 0$ show that the critical value of ω_0 required to flip the frame (and ring) into the stationary vertical configuration is

$$
\omega_0 = \sqrt{\left(\frac{5}{6} ML^2 + m(R^2 + \frac{L^2}{2}) \right)(m+M)} \frac{\sqrt{gL}\sqrt{\left(\sqrt{2}-1\right)}}{mR^2}
$$

Energy is conserved as the frame rotates up onto its edge.

The formula for the kinetic energy of a system of rigid bodies is

$$T = \sum \frac{1}{2} m \left| \mathbf{v}_G \right|^2 + \sum \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}_G \boldsymbol{\omega}$$

For 2D problems we can replace the last term by $\frac{1}{2} I_{Gzz} \omega_z^2$

Assume that the frame is at rest in the upright state. The total potential and kinetic energy in the upright state is therefore

$$T_1 + U_1 = \frac{1}{2} m R^2 \omega_1^2 + (m+M) g \frac{L}{\sqrt{2}}$$

In the initial state

$$\begin{aligned} T_0 + U_0 &= \frac{1}{2}(m+M)\left(\frac{L}{\sqrt{2}}\omega_f\right)^2 + \frac{1}{2} m R^2 \left(\omega_1 + \omega_f\right)^2 + \frac{1}{2}\frac{1}{3} M L^2 \omega_f^2 + (m+M) g \frac{L}{2} \\ &= \frac{1}{2}\left(\frac{5}{6} M L^2 + m(R^2 + \frac{L^2}{2})\right)\omega_f^2 + m R^2 \omega_1 \omega_f + \frac{1}{2} m R^2 \omega_1^2 + (m+M) g \frac{L}{2} \end{aligned}$$

Energy conservation gives

$$\frac{1}{2}\left(\frac{5}{6} M L^2 + m(R^2 + \frac{L^2}{2})\right)\omega_f^2 + m R^2 \omega_1 \omega_f + \frac{1}{2} m R^2 \omega_1^2 + (m+M) g \frac{L}{2} = \frac{1}{2} m R^2 \omega_1^2 + (m+M) g \frac{L}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{2}\left(\frac{5}{6} M L^2 + m(R^2 + \frac{L^2}{2})\right)\omega_f^2 + m R^2 \omega_1 \omega_f - (m+M) g L \left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = 0$$

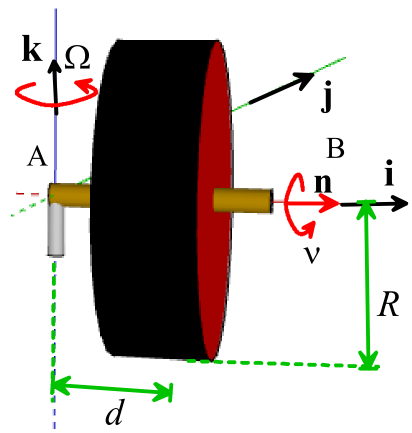
For $\omega_1 = 0$ we get

$$\omega_f = \sqrt{\frac{(m+M)}{\left(\frac{5}{6} M L^2 + m(R^2 + \frac{L^2}{2})\right)}} \sqrt{gL} \sqrt{\left(\sqrt{2} - 1\right)}$$

From part 4 we get

$$\omega_0 = \frac{\left(\frac{5}{6} M L^2 + m(R^2 + \frac{L^2}{2})\right)}{m R^2} \omega_f = \sqrt{\left(\frac{5}{6} M L^2 + m(R^2 + \frac{L^2}{2})\right)(m+M)} \frac{\sqrt{gL} \sqrt{\left(\sqrt{2} - 1\right)}}{m R^2}$$

6.8.2 Solutions to 3D problems



Example 1: The figure shows a wheel spinning on a frictionless axle. The axle is supported on one side (at A) by a pivot that allows free rotation in any direction. If the wheel were not spinning, it would simply swing about A like a pendulum. But if the angular speed is high enough, the axle remains horizontal, and the wheel turns slowly about the vertical axis. This behavior is called 'precession' and is a bit mysterious – why does spin somehow hold the wheel up? The goal of this example is to explain this, and to calculate a formula for the rotation rate of the axle.

We will do this by showing that steady precession satisfies all the equations of motion.

1.1 Let **n** be a unit vector parallel to the axle. Consider the disk at the instant when $\mathbf{n}=\mathbf{i}$, and assume that

- the disk spins at constant rate about the axle at ν radians per second,
- the disk rotates slowly at constant rate about **k** at Ω radians per second

Find the angular velocity and angular acceleration at the instant shown in the figure

The angular velocity is easy – we just add the two vectors: $\boldsymbol{\omega}=\nu\mathbf{n}+\Omega\mathbf{k}=\nu\mathbf{i}+\Omega\mathbf{k}$

The angular acceleration is harder. Both ω and Ω are constant. But this does not mean that the angular velocity **vector** is constant, because the axle is rotating about the **k** axis. The *direction* of the angular velocity is changing, even though the magnitude is not. We can calculate the rate of change of **n** by using the rigid body formula

$$\frac{d}{dt}(\mathbf{r}_B-\mathbf{r}_A)=\mathbf{v}_B-\mathbf{v}_A=\boldsymbol{\omega}\times(\mathbf{r}_B-\mathbf{r}_A)$$

If we choose A and B to be a unit distance apart, then $(\mathbf{r}_B-\mathbf{r}_A)=\mathbf{n}$ and therefore

$$\frac{d\mathbf{n}}{dt}=\boldsymbol{\omega}\times\mathbf{n}$$

We can now calculate the angular acceleration

$$\boldsymbol{\alpha}=\frac{d\boldsymbol{\omega}}{dt}=\frac{d}{dt}(\Omega\mathbf{k}+\nu\mathbf{n})=\nu\frac{d\mathbf{n}}{dt}=\nu(\Omega\mathbf{k}+\nu\mathbf{n})\times\mathbf{n}=\nu\Omega\mathbf{j}$$

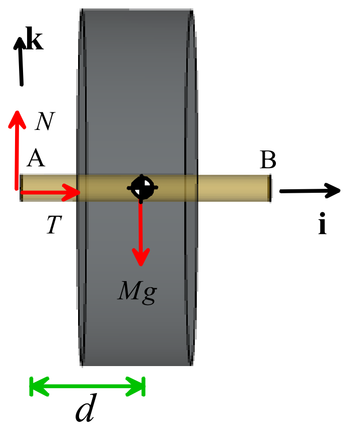
1.2 Find a formula for the acceleration of the center of mass of the disk

We can use the rigid body formula

$$\begin{aligned}\mathbf{a}_G-\mathbf{a}_A&=\boldsymbol{\alpha}\times(\mathbf{r}_G-\mathbf{r}_A)+\boldsymbol{\omega}\times\left[\boldsymbol{\omega}\times(\mathbf{r}_G-\mathbf{r}_A)\right]\\&=\nu\Omega\mathbf{j}\times(d\mathbf{i})+\left(\nu\mathbf{i}+\Omega\mathbf{k}\right)\times\left[\left(\nu\mathbf{i}+\Omega\mathbf{k}\right)\times(d\mathbf{i})\right]\\&=-\Omega^2 d\mathbf{i}\end{aligned}$$

A quicker way is to notice that the COM is in circular motion around A and use the circular motion formula, with the same result.

1.3 Draw a free body diagram showing the forces acting on the wheel



1.3 Write down the equations of translational and rotational motion for the disk

$$\sum_i \mathbf{F}^{(i)} = M\mathbf{a}_G \Rightarrow T\mathbf{i} + (N - Mg)\mathbf{k} = -Md\Omega^2\mathbf{i}$$

$$\sum_i \mathbf{r}_i \times \mathbf{F}^{(i)} + \sum_j \mathbf{Q}^{(j)} = M\mathbf{r}_G \times \mathbf{a}_G + \mathbf{I}_G\boldsymbol{\alpha} + \boldsymbol{\omega} \times \left[\mathbf{I}_G\boldsymbol{\omega}\right]$$

$$\Rightarrow d\mathbf{i} \times (-Mg\mathbf{k}) = M(d\mathbf{i}) \times (-\Omega^2 d\mathbf{i}) + \begin{bmatrix} I_{Gxx} & 0 & 0 \\ 0 & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix} \begin{bmatrix} 0 \\ \nu\Omega \\ 0 \end{bmatrix} + \begin{bmatrix} \nu \\ 0 \\ \Omega \end{bmatrix} \times \begin{bmatrix} I_{Gxx} & 0 & 0 \\ 0 & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix} \begin{bmatrix} \nu \\ 0 \\ \Omega \end{bmatrix}$$

Working through the cross products and the matrix-vector products we get

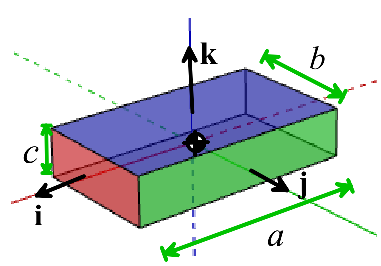
$$Mgd\mathbf{j} = I_{Gyy}\nu\Omega\mathbf{j} - (I_{Gzz} - I_{Gxx})\nu\Omega\mathbf{j}$$

We see that steady precession can indeed satisfy all the equations of motion. Moreover, for a disk (or any solid of revolution) $I_{Gzz} = I_{Gxx}$, so we can calculate the precession rate

$$Mgd\mathbf{j} = I_{Gxx}\nu\Omega\mathbf{j}$$

$$\Rightarrow \Omega = \frac{Mgd}{I_{Gxx}\nu}$$

Example 2: The prism shown in the figure floats in space (no gravity). At time $t=0$ its faces are perpendicular to the **i,j,k** axes as shown. It is then given an initial angular velocity $\boldsymbol{\omega} = \omega_{z0}\mathbf{k} + \omega_{y0}\mathbf{j}$ with $\omega_{y0} << \omega_{z0}$ (i.e. we set the body spinning about the **k** axis, but give it a very small disturbance) . Investigate the nature of the subsequent motion, with both hand calculations and by writing a MATLAB script that will animate the motion of the prism.



No forces or moments act on the prism. We can use the equations of motion

$$\mathbf{0} = M\mathbf{a}_G \qquad \mathbf{0} = M\mathbf{r}_G \times \mathbf{a}_G + \mathbf{I}_G\boldsymbol{\alpha} + \boldsymbol{\omega} \times [\mathbf{I}_G\boldsymbol{\omega}]$$

The angular momentum equation can be written out explicitly

$$\begin{bmatrix} I_{Gxx} & 0 & 0 \\ 0 & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix} \begin{bmatrix} d\omega_x / dt \\ d\omega_y / dt \\ d\omega_z / dt \end{bmatrix} + \left[\omega_x, \omega_y, \omega_z\right] \times \left(\begin{bmatrix} I_{Gxx} & 0 & 0 \\ 0 & I_{Gyy} & 0 \\ 0 & 0 & I_{Gzz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \right)$$

(we could substitute values for $I_{Gxx}, I_{Gyy}, I_{Gzz}$ in terms of *a,b,c* and *M* but it is clearer to leave them) Expanding out the matrix products and cross product gives

$$I_{Gxx}\frac{d\omega_x}{dt} + \left(I_{Gzz} - I_{Gyy}\right)\omega_y\omega_z = 0$$

$$I_{Gyy}\frac{d\omega_y}{dt} - \left(I_{Gzz} - I_{Gxx}\right)\omega_x\omega_z = 0$$

$$I_{Gzz}\frac{d\omega_z}{dt} + \left(I_{Gyy} - I_{Gxx}\right)\omega_x\omega_y = 0$$

At time *t=0* ω_x is zero and ω_y is small. They might increase, but we will only consider behavior while they remain small. In this case $\omega_x\omega_y$ is extremely small so we can assume $d\omega_z / dt \approx 0$. We can then decouple the first two equations like this:

1. Differentiate the second equation with respect to time $I_{Gyy}\dfrac{d^2\omega_y}{dt^2} - \left(I_{Gzz} - I_{Gxx}\right)\dfrac{d\omega_x}{dt}\omega_{z0} = 0$
2. Now we can substitute for $d\omega_x / dt$ using the first equation, and divide by I_{Gyy}

$$\frac{d^2\omega_y}{dt^2} + \frac{\left(I_{Gzz} - I_{Gxx}\right)\left(I_{Gzz} - I_{Gyy}\right)}{I_{Gyy}I_{Gxx}}\omega_{z0}^2\omega_y = 0$$

This is an equation of the form

$$\frac{d^2\omega_y}{dt^2} + \lambda\omega_y = 0$$

We recognize this as an undamped vibration equation (case I or case II from our table of solutions). Its solution depends on the sign of λ :

1. For $\lambda > 0$ the solution is $\omega_y = A\sin\sqrt{\lambda}t + B\cos\sqrt{\lambda}t$ where *A, B* are constants. **This is stable motion** - ω_y remains small.
2. For $\lambda < 0$ the solution is $\omega_y = A\exp\sqrt{\lambda}t + B\exp(-\sqrt{\lambda}t)$. **This is unstable motion -** ω_y will become very large.

The sign of λ is determined by the product $\left(I_{Gzz} - I_{Gxx}\right)\left(I_{Gzz} - I_{Gyy}\right)$. There are three possible cases:

1. I_{Gzz} is greater than I_{Gxx}, I_{Gyy} (the **k** axis has the maximum inertia). Motion is stable
2. I_{Gzz} is less than I_{Gxx}, I_{Gyy} (the **k** axis has the minimum inertia). Motion is stable
3. I_{Gzz} is between I_{Gxx}, I_{Gyy} . **Motion is unstable**.

We can learn more about the motion by using MATLAB to solve the equations of motion for us. Since there is no motion of the center of mass, we only need to consider rotational motion. We know that we can describe the orientation of the prism by the rotation tensor $\mathbf{R}$ and its rate of change of orientation by the angular velocity $\boldsymbol{\omega}$. The orientation and angular velocity are governed by the differential equations

$$\frac{d\mathbf{R}}{dt} = \mathbf{W}\mathbf{R}$$

$$\mathbf{I}_G \frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \times \left[\mathbf{I}_G \boldsymbol{\omega} \right] = \mathbf{0}$$

where $\mathbf{I}_G = \mathbf{R}\mathbf{I}_G^0\mathbf{R}^T$ is the rotated inertia tensor for the block, and $\mathbf{W}$ is the spin tensor

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

We need to set up the MATLAB 'ode' solver to calculate $\mathbf{R}$ and $\boldsymbol{\omega}$ as functions of time by integrating these equations.

We can store the unknown rotation matrix and the angular velocity vector in a MATLAB vector:

$$\mathbf{w} = \left[R_{xx}, R_{xy}, R_{xz}, R_{yx}, R_{yy}, R_{yz}, R_{zx}, R_{zy}, R_{zz}, \omega_x, \omega_y, \omega_z \right]$$

We need to write a MATLAB function that will calculate the time derivatives of this vector, given its current value. The calculation involves the following steps:

(1) Assemble the vectors $\boldsymbol{\omega}$ and the rotation tensor $\mathbf{R}$ from the Matlab solution vector **w.** Matlab has a useful function that will automatically convert a matrix to a vector, and vice-versa. For example, $\mathbf{R}$ (a 3x3 matrix) can be converted to **w** (a 1x9 column vector) using

```
w = reshape(transpose(R),[9,1]))
```

To transform **w** (as a column vector) back to **R**, you can use

```
R = transpose(reshape(w,[3,3]))
```

(2) Calculate the spin tensor **W**

(3) Calculate the rotated inertia tensor $\mathbf{I}_G = \mathbf{R}\mathbf{I}_G^0\mathbf{R}^T$ (Matlab will multiply the matrices for us)

(4) Solve the equations for the angular acceleration $\boldsymbol{\alpha}$

(5) Calculate $d\mathbf{R}/dt = \mathbf{W}\mathbf{R}$

(6) Assemble the matlab vector $\frac{d\mathbf{w}}{dt} = \left[\dot{R}_{xx}, \dot{R}_{xy}, \dot{R}_{xz}, \dot{R}_{yx}, \dot{R}_{yy}, \dot{R}_{yz}, \dot{R}_{zx}, \dot{R}_{zy}, \dot{R}_{zz}, \alpha_x, \alpha_y, \alpha_z \right]$

This sounds complicated but actually MATLAB is great at doing this sort of calculation efficiently. Here's a function:

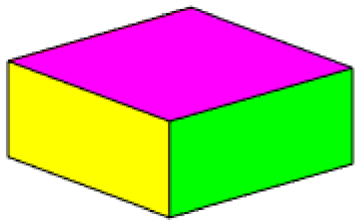
```
function  dwdt = rigid_body_eom(t,w)
   Rvec = w(1:9); % Rotation matrix, stored as a vector
   omega = w(10:12);  % Angular velocity
   R = transpose(reshape(Rvec,[3,3]));
   II = R*I0*transpose(R); %Current inertia tensor, in fixed coord system
   W = [0,-omega(3),omega(2);omega(3),0,-omega(1);-omega(2),omega(1),0];
   alpha = -II\(cross(omega,II*omega)); % Angular accel
   Rdot = W*R; % Rate of change of rotation matrix
   Rdotvec = reshape(transpose(Rdot),[9,1]);
   dwdt = [Rdotvec;alpha];
end
```

We just need to set up ode45 to integrate (numerically) the differential equation:

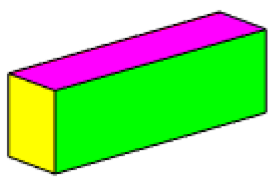
```
omega0 = [0.,0.01,1];    % Initial angular velocity
a = [4,1,2];             % Dimensions (a,b,c) of the prism
time = 20;
initial_w = [1;0;0;0;1;0;0;0;1;transpose(omega0(1:3))];
I0 = [a(2)^2+a(3)^2,0,0;0,a(1)^2+a(3)^2,0;0,0,a(1)^2+a(2)^2];
options = odeset('RelTol',0.00000001);
sol = ode45(@(t,w) rigid_body_eom(t,w,I0),[0,time],initial_w,options);
animate_rigid_body(sol,a,[0,time])
```

You can download the full script here.

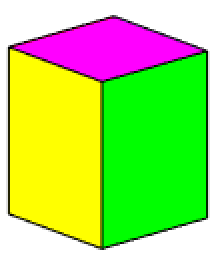
The figures below show animations of the predicted behavior for the three possible types of behavior



I_{zz} is the maximum inertia – rotation is stable



I_{zz} is the intermediate inertia – rotation is unstable (the block tumbles)



I_{zz} is the smallest inertia – rotation is stable